On the distribution of the trace in the unitary symplectic group and the distribution of Frobenius

Gilles Lachaud

ON THE DISTRIBUTION OF THE TRACE IN THE UNITARY SYMPLECTIC GROUP AND THE DISTRIBUTION OF FROBENIUS

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Abstract. The purpose of this article is to study the distribution of the trace on the unitary symplectic group. We recall its relevance to equidistribution results for the eigenvalues of the Frobenius in families of abelian varieties over finite fields, and to the limiting distribution of the number of points of curves. We give four expressions of the trace distribution if $g = 2$, in terms of special functions, and also an expression of the distribution of the trace in terms of elementary symmetric functions. In an appendix, we prove a formula for the trace of the exterior power of the identity representation.

Contents

1. Introduction

Let G be a connected compact Lie group, and $\pi : G \longrightarrow \mathbf{GL}(V)$ a continuous representation of G on a finite dimensional complex vector space V. The map

$$m \mapsto \boldsymbol{\tau}(m) = \operatorname{Trace} \pi(m)$$

is a continuous central function on G, whose values lie in a compact interval $I \subset \mathbb{R}$. The *distribution* or *law* of $\boldsymbol{\tau}$ is the measure $\mu_{\boldsymbol{\tau}} = \boldsymbol{\tau}_*(dm)$ on I which is the image

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by τ of the mass one Haar measure dm on G. That is, for any continuous real function $\varphi \in \mathscr{C}(I)$, we impose the integration formula

$$\int_I \varphi(x)\, \mu_{\boldsymbol{\tau}}(x) = \int_G \varphi(\operatorname{Trace} \pi(m))\, dm.$$

Alternately, if $x \in \mathbb{R}$, then

$$\text{volume}\, \{m \in G \mid \operatorname{Trace} \pi(m) \leq x\} = \int_{-\infty}^{x} \mu_{\boldsymbol{\tau}}.$$

We are especially interested here with the group $G = \mathbf{USp}_{2g}$ of symplectic unitary matrices of order $2g$, with π equal to the identity representation in $V = \mathbb{C}^4$. With the help of Weyl's integration formula, one establishes that the distribution $\mu_{\boldsymbol{\tau}}$ has a *density* $f_{\boldsymbol{\tau}}$, that is, a positive continuous function such that $d\mu_{\boldsymbol{\tau}}(x) = f_{\boldsymbol{\tau}}(x)dx$. Our main purpose is the study of $f_{\boldsymbol{\tau}}$, especially in the case $g = 2$ and $g = 3$. For instance, for $g = 2$, we have

$$f_{\boldsymbol{\tau}}(x) = \frac{1}{4\pi}\left(1 - \frac{x^2}{16}\right)^4 {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 5; 1 - \frac{x^2}{16}\right)$$

if $|x| \leq 4$, with *Gauss' hypergeometric function* ${}_2F_1$ (see Theorem 5.2).

Another representation of the distribution of the trace, following a program of Kohel, is realized by the *Viète map*, which is the polynomial mapping

$$\mathsf{s}(t) = (s_1(t), \ldots, s_g(t)), \quad t = (t_1, \ldots, t_g),$$

where $s_n(t)$ is the elementary symmetric polynomial of degree n. Let $I_g = [-2, 2]^g$. The *symmetric alcove* is the set

$$\Sigma_g = \mathsf{s}(I_g) \subset \mathbb{R}^g,$$

which is homeomorphic to the g-dimensional simplex. By a change of variables in Weyl's integration formula, one obtains a measure α_x on the hyperplane section

$$V_x = \{s \in \Sigma_g \mid s_1 = x\}$$

such that, if $|x| < 2g$,

$$f_{\boldsymbol{\tau}}(x) = \int_{V_x} \alpha_x(s).$$

As a motivation for the study of these distributions, it is worthwhile to recall that they provide an answer to the following question:

Can one predict the number of points of a curve of given genus over a finite field?

When a curve C runs over the set $\mathsf{M}_g(\mathbb{F}_q)$ of $\mathbb{F}_q$-isomorphism classes of (nonsingular, absolutely irreducible) curves of genus g over $\mathbb{F}_q$, the number $|C(\mathbb{F}_q)|$ seems to vary at random. According to Weil's inequality, an accurate approximation to this number is close to $q + 1$, with a normalized "error" term $\boldsymbol{\tau}(C)$ such that

$$|C(\mathbb{F}_q)| = q + 1 - q^{1/2}\, \boldsymbol{\tau}(C), \quad |\boldsymbol{\tau}(C)| \leq 2g.$$

The random matrix model developed by Katz and Sarnak gives many informations on the behaviour of the distribution of $\boldsymbol{\tau}(C)$ on the set $\mathsf{M}_g(\mathbb{F}_q)$. For instance, according to their theory, and letting g be fixed, for every $x \in \mathbb{R}$, we have, as $q \to \infty$ (cf. Corollary 4.3):

$$\frac{|\{C \in \mathsf{M}_g(\mathbb{F}_q) \mid \boldsymbol{\tau}(C) \leq x\}|}{|\mathsf{M}_g(\mathbb{F}_q)|} = \int_{-\infty}^{x} f_{\boldsymbol{\tau}}(s)ds + O\left(q^{-1/2}\right).$$

Hence, the knowledge of $f_{\boldsymbol{\tau}}$ provides a precise information on the behaviour of the distribution of the number of points of curves.

The outline of this paper is as follows. After Section 2, devoted to notation, we recall in Section 3 the Weyl's integration formula, expressed firstly in terms of the angles $(\theta_1 \ldots, \theta_g)$ defining a conjugacy class, and secondly in terms of the coefficients $t_j = 2\cos\theta_j$. We discuss equidistribution results for a family of curves or abelian varieties over a finite field in Section 4. In Section 5 we obtain four explicit formulas for the trace distribution if $g = 2$, respectively in terms of hypergeometric series, of Legendre functions, of elliptic integrals, and of Meijer G-functions. We also give the distribution of the trace for the representation of the group $\mathbf{SU}_2 \times \mathbf{SU}_2$ in $\mathbf{USp}_4$.

In the second part of the paper, we take on a different point of view by using elementary symmetric polynomials, and obtain a new expression of Weyl's integration formula. Section 6 defines the Viète map, asssociating to a sequence of coordinates the coefficients of the polynomial admitting as roots the elements of this sequence, and Section 7 describes the symmetric alcove, that is, the image of the set of "normalized real Weil polynomials" by the Viète map. By a change of variables using the Viète map, we obtain in Section 8 a new integration formula, which leads to another expression for the distribution of the trace on the conjugacy classes, in the cases $g = 2$ and $g = 3$. If $g = 2$, we compute also the trace of $\wedge^2\pi$. Finally, we include an appendix on the character ring of $\mathbf{USp}_{2g}$, including a formula on the exterior powers of the identity representation.

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2. The unitary symplectic group

The *unitary symplectic group* $G = \mathbf{USp}_{2g}$ of order $2g$ is the real Lie group of complex symplectic matrices

$$G = \left\{ m \in \mathbf{GL}_{2g}(\mathbb{C}) \mid {}^t m.J.m = J \text{ and } {}^t m.\bar{m} = \mathbf{I}_{2g} \right\},$$

with

$$J = \begin{pmatrix} 0 & \mathbf{I}_g \\ -\mathbf{I}_g & 0 \end{pmatrix}.$$

Alternately, the elements of G are the matrices

$$m = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in \mathbf{SU}_{2g}, \quad a, b \in \mathbf{M}_g(\mathbb{C}).$$

The torus $\mathbb{T}^g = (\mathbb{R}/2\pi\mathbb{Z})^g$ is embedded into G by the homomorphism

$$\theta = (\theta_1, \ldots, \theta_g) \mapsto h(\theta) = \begin{pmatrix} \begin{matrix} e^{i\theta_1} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & e^{i\theta_g} \end{matrix} & 0 \\ 0 & \begin{matrix} e^{-i\theta_1} & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & e^{-i\theta_g} \end{matrix} \end{pmatrix} \tag{2.1}$$

whose image T is a maximal torus in G. The Weyl group W of (G, T) is the semi-direct product of the symmetric group $\mathfrak{S}_g$ in g letters, operating by permutations on the θ_j, and of the group N of order 2^g generated by the involutions $\theta_j \mapsto -\theta_j$. Since every element of $\mathbf{USp}_{2g}$ has eigenvalues consisting of g pairs of complex conjugate numbers of absolute value one, the quotient T/W can be identified with the set $\mathrm{Cl}\, G$ of conjugacy classes of G, leading to a homeomorphism

$$\mathbb{T}^g/W \xrightarrow{\sim} T/W \xrightarrow{\sim} \mathrm{Cl}\, G$$

Remark 2.1. Here is a simple description of the set $\mathrm{Cl}\, G$. Let Φ_{2g} be the subset of monic polynomials $p \in \mathbb{R}[u]$ of degree $2g$, with $p(0) = 1$, with roots consisting of g pairs of complex conjugate numbers of absolute value one. If $\theta \in \mathbb{T}^g$, let

$$p_\theta(u) = \prod_{j=1}^{g} (u - e^{i\theta_j})(u - e^{-i\theta_j}).$$

The map $\theta \mapsto p_\theta$ is a bijection from $\mathbb{T}^g/W$ to Φ_{2g}. Renumbering, we may assume that

$$0 \le \theta_g \le \theta_{g-1} \le \cdots < \theta_1 \le \pi.$$

The map $m \mapsto \mathrm{cp}_m(u) = \det(u.\mathbf{I} - m)$ induces a homeomorphism

$$\mathrm{Cl}\, G \xrightarrow{\sim} \Phi_{2g}$$

with $\mathrm{cp}_m = p_\theta$ if and only if m is conjugate to $h(\theta)$. The polynomial p_θ is *palindromic*, that is, if

$$p_\theta(u) = \sum_{n=0}^{2g} (-1)^n a_n(\theta) u^n,$$

then $a_{2g-n}(\theta) = a_n(\theta)$ for $0 \le n \le g$.

3. Weyl's integration formula

The box $X_g = [0, \pi]^g$ is a fundamental domain for N in $\mathbb{T}^g$ and the map $F \mapsto F \circ h$ defines an isomorphism

$$\mathscr{C}(G)^\circ \xrightarrow{\sim} \mathscr{C}(X_g)^{\mathrm{sym}} \tag{3.1}$$

from the vector space $\mathscr{C}(G)^\circ = \mathscr{C}(\mathrm{Cl}\, G)$ of *complex central continuous functions* on G to the space $\mathscr{C}(X_g)^{\mathrm{sym}}$ of complex symmetric continuous functions on X_g. Notice that the isomorphism (3.1) has an algebraic analog, namely the isomorphism (A.1) in the appendix. Let dm be the Haar measure of volume 1 on G. If $\mathsf{F} \in \mathscr{C}(G)^\circ$, then

$$\int_{\mathrm{Cl}\, G} \mathsf{F}(\dot{m})\, d\dot{m} = \int_G \mathsf{F}(m)\, dm,$$

where $d\dot{m}$ is the image measure on $\mathrm{Cl}\, G$ of the measure dm. The following result is classical [4, Ch. 9, § 6, Th. 1, p. 337], [10, 5.0.4, p. 107].

Theorem 3.1 (Weyl integration formula, I)**.** *If* $\mathsf{F} \in \mathscr{C}(G)^\circ$*, then*

$$\int_G \mathsf{F}(m)\, dm = \int_{X_g} \mathsf{F} \circ h(\theta))\mu_g(\theta),$$

with the Weyl measure

$$\mu_g(\theta) = \delta_g(\theta) d\theta, \quad d\theta = d\theta_1 \dots d\theta_g,$$

$$\delta_g(\theta) = \frac{1}{g!} \prod_{j=1}^{g} \left(\frac{2}{\pi}\right) (\sin\theta_j)^2 \prod_{j<k} (2\cos\theta_k - 2\cos\theta_j)^2 . \quad \square$$

We call the open simplex

$$A_g = \{(\theta_1, \dots, \theta_g) \in X_g \mid 0 < \theta_g < \theta_{g-1} < \cdots < \theta_1 < \pi\} \tag{3.2}$$

the *fundamental alcove* in X_g. The closure $\bar{A}_g$ of A_g is a fundamental domain for $\mathfrak{S}_g$ in X_g, and, for every $f \in \mathscr{C}(X_g)^{\mathrm{sym}}$, we have

$$\int_{X_g} f(\theta)\, d\theta = g! \int_{A_g} f(\theta)\, d\theta.$$

There is another way to state Weyl's integration formula, which will be used in Section 8. Let $I_g = [-2, 2]^g$. The map

$$(\theta_1, \dots, \theta_g) \mapsto (2\cos\theta_1, \dots, 2\cos\theta_g)$$

defines an homeomorphism $X_g \longrightarrow I_g$. Let

$$k(t_1, \dots, t_g) = h\left(\arccos\frac{t_1}{2}, \dots, \arccos\frac{t_g}{2}\right).$$

where $h(\theta)$ is given by (2.1). Then the map $F \mapsto F \circ k$ defines an isomorphism

$$\mathscr{C}(G)^\circ \xrightarrow{\sim} \mathscr{C}(I_g)^{\mathrm{sym}}$$

where $\mathscr{C}(I_g)^{\mathrm{sym}}$ is the space of complex symmetric continuous function on I_g. For an algebraic analog, see the isomorphism (A.2) in the appendix. Let

$$D_0(t) = \prod_{j<k} (t_k - t_j)^2, \quad D_1(t) = \prod_{j=1}^{g} (4 - t_j^2). \tag{3.3}$$

Proposition 3.2 (Weyl integration formula, II). *If $\mathsf{F} \in \mathscr{C}(G)^\circ$, then*

$$\int_G \mathsf{F}(m)\, dm = \int_{I_g} \mathsf{F} \circ k(t) \lambda_g(t)\, dt,$$

where $t = (t_1, \dots, t_g)$ and $dt = dt_1 \dots dt_g$, with the Weyl measure

$$\lambda_g(t) dt, \quad dt = dt_1 \dots dt_g,$$

$$\lambda_g(t) = \frac{1}{(2\pi)^g g!} D_0(t) \sqrt{D_1(t)}.$$

Proof. If $\varphi \in \mathscr{C}(I_g)$, we have

$$\int_{X_g} \varphi(2\cos\theta_1, \dots, 2\cos\theta_g) \delta_g(\theta)\, d\theta = \int_{I_g} \varphi(t) \lambda_g(t) dt.$$

Apply Weyl's integration formula of Theorem 3.1. □

As in (3.2), we call the open simplex

$$A_g = \{t \in I_g \mid -2 < t_1 < t_2 < \cdots < t_g < 2\} \tag{3.4}$$

the *fundamental alcove in* I_g. Then $\bar{A}_g$ is a fundamental domain of I_g for $\mathfrak{S}_g$, and if $f \in \mathscr{C}(I_g)^{\mathrm{sym}}$, we have

$$\int_{I_g} f(t)\, dt = g! \int_{A_g} f(t)\, dt. \tag{3.5}$$

Examples 3.3. We have

$$\lambda_2(t) = \frac{1}{4\pi^2}(t_1 - t_2)^2\sqrt{(4 - t_1^2)(4 - t_2^2)}.$$

The maximum of λ_2 in A_2 is attained at the point

$$t_0 = (-\sqrt{2}, \sqrt{2}), \quad \text{with} \quad \lambda_2(t_0) = \frac{2}{\pi^2}\,.$$

We have also

$$\lambda_3(t) = \frac{1}{48\pi^3}(t_1 - t_2)^2(t_1 - t_3)^2(t_2 - t_3)^2\sqrt{(4 - t_1^2)(4 - t_2^2)(4 - t_3^2)}.$$

The maximum of λ_3 in A_3 is attained at the point

$$t_0 = (-\sqrt{3}, 0, \sqrt{3}), \quad \text{with} \quad \lambda_3(t_0) = \frac{9}{2\pi^3}\,.$$

Now, for the convenience of the reader, we recall some notation on the distribution of central functions. Let G be a connected compact Lie group. The Haar measure dm of volume 1 on G is a probability measure, and G becomes a probability space; *ipso facto*, its elements become random matrices, and the functions in $\mathscr{C}(G)^\circ$ are complex random variables on G. If $\mathsf{F} \in \mathscr{C}(G)^\circ$ is a real random variable, whose values lie in the compact interval $I \subset \mathbb{R}$, the *distribution* or *law* of F is the image measure $\mu_{\mathsf{F}} = F_* dm$ on I such that

$$\int_I \varphi(x)\,\mu_{\mathsf{F}}(x) = \int_G \varphi(\mathsf{F}(m))\,dm \quad \text{if} \quad \varphi \in \mathscr{C}(I),$$

If B is a borelian subset of I, then

$$\mu_{\mathsf{F}}(B) = \text{volume}\,\{m \in G \mid \mathsf{F}(m) \in B\}\,,$$

and the *cumulative distribution function* of F is

$$\Phi_{\mathsf{F}}(x) = \mathbb{P}(\mathsf{F} \le x) = \int_{-\infty}^x \mu_{\mathsf{F}}(t) = \int_{\mathsf{F}(m) \le x} dm.$$

The *characteristic function* of F is the Fourier transform of $\mu_{\mathsf{F}}(x)$:

$$\widehat{f}_{\mathsf{F}}(t) = \int_{-\infty}^{\infty} e^{itx}\mu_{\mathsf{F}}(x) = \int_G e^{it\mathsf{F}(m)}\,dm = \int_{X_g} e^{it\mathsf{F}\circ h(\theta)}\mu_g(\theta).$$

This is an entire analytic function of t, of exponential type, bounded on the real line. The distribution μ_{F} has a *density* if $\mu_{\mathsf{F}}(x) = f_{\mathsf{F}}(x)dx$ with a positive function f_{F} in $L^1(I)$. If μ_{F} has a density, and if Fourier inversion holds, then

$$f_{\mathsf{F}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}_{\mathsf{F}}(t)e^{-itx}\,dt.$$

Conversely, if $\widehat{f}_{\mathsf{F}} \in L^1(\mathbb{R})$, then μ_{F} has a density. If $G = \mathbf{USp}_{2g}$, notice that Weyl's integration formulas supply the *joint probability density function* for the random variables $(\theta_1, \dots, \theta_g)$ and $(t_1, \dots, t_g)$.

The distribution μ_{F} is characterized by the sequence of its *moments*

$$M_n(\mathsf{F}) = \int_I x^n\,\mu_{\mathsf{F}}(x) = \int_G \mathsf{F}(m)^n\,dm, \qquad n \ge 1,$$

and the characteristic function is a generating function for the moments:

$$\widehat{f}_{\mathsf{F}}(t) = \sum_{n=0}^{\infty} M_n(\mathsf{F})\frac{(it)^n}{n!}\,. \tag{3.6}$$

Remark 3.4. If π is an irreducible representation of G, with real character $\boldsymbol{\tau}_\pi$, then the random variable $\boldsymbol{\tau}_\pi$ is *standardized*, *i.e.* the first moment (the mean) is equal to zero and the second moment (the variance) is equal to one.

Remark 3.5. Under suitable conditions, an expression of the density by integration along the fibers can be given. For instance, let $G = \mathbf{USp}_{2g}$, let $\mathsf{F} \in \mathscr{C}(G)^\circ$ be a C^∞ function, and put $J = \mathsf{F} \circ h(U)$, where U is the open box $]0, \pi[^g$. If $\mathsf{F} \circ h$ is a submersion on U, and if $x \in J$, then

$$V_x = \{\theta \in U \mid \mathsf{F} \circ h(\theta) = x\}$$

is a hypersurface. Let α_x be the *Gelfand-Leray differential form* on V_x, defined by the relation

$$d(\mathsf{F} \circ h) \wedge \alpha_x = \delta_g(\theta) d\theta_1 \wedge \cdots \wedge d\theta_g.$$

For instance,

$$\alpha_x = (-1)^{j-1}(\partial(\mathsf{F} \circ h)/\partial\theta_j)^{-1}\delta_g(\theta) d\theta_1 \wedge \dots d\theta_{j-1} \wedge d\theta_{j+1} \cdots \wedge d\theta_g$$

if the involved partial derivative is $\neq 0$. Then the distribution is computed by slicing: since the cumulative distribution function is

$$\Phi_{\mathsf{F}}(x) = \int_{\mathsf{F}\circ h(\theta) \leq x} \mu_g(\theta) = \int_{-\infty}^{x} ds \int_{V_s} \alpha_s(\theta),$$

we have

$$f_{\mathsf{F}}(x) = \int_{V_x} \alpha_x(\theta).$$

See [1, Lemma 7.2] and [17, Lemma 8.5].

4. Equidistribution

Let A be an abelian variety of dimension g over $\mathbb{F}_q$. The *Weil polynomial* of A is the characteristic polynomial $L(A, u) = \det(u.\mathbf{I} - F_A)$ of the Frobenius endomorphism F_A of A, and the *unitarized Weil polynomial* of A is

$$\bar{L}(A, u) = L(A, q^{-1/2}u) = \prod_{j=1}^{g}(u - e^{i\theta_j})(u - e^{-i\theta_j}).$$

This polynomial has coefficients in $\mathbb{Z}$, belongs to the set Φ_{2g} defined in Remark 2.1, and $\theta(A) = (\theta_1, \dots \theta_g)$ is the *sequence of Frobenius angles* of A. We write

$$\bar{L}(A, u) = \sum_{n=0}^{2g}(-1)^n a_n(A) u^n,$$

keeping in mind that $a_{2g-n}(A) = a_n(A)$ for $0 \leq n \leq 2g$, since $\bar{L}(A, u) \in \Phi_{2g}$. By associating to A the polynomial $\bar{L}(A, u)$, each abelian variety defines, as explained in Section 2, a unique class $\dot{m}(A)$ in $\mathrm{Cl}\, G$, such that

$$\bar{L}(A, u) = \det(u\,\mathbf{I} - \dot{m}(A)).$$

Let $\mathsf{A}_g(\mathbb{F}_q)$ be the finite set of k-isomorphism classes of principally polarized abelian varieties of dimension g over k. The following question naturally arises:

As $q \to \infty$, and as A runs over $\mathsf{A}_g(\mathbb{F}_q)$, what are the limiting distributions of the random variables $a_1, \dots, a_g$?

In order to clarify this sentence, we look in particular to the coefficient a_1, and focus on the Jacobians of curves. Let C be a (nonsingular, absolutely irreducible, projective) curve over $\mathbb{F}_q$. The *Weil polynomial* $L(C,u)$ of C is the Weil polynomial of its Jacobian, and similarly for the *unitarized Weil polynomial* $\bar{L}(C,u)$, the *sequence of Frobenius angles* $\theta(C)$, the coefficients $a_n(C)$, and the conjugacy class $\dot{m}(C)$. If F_C is the geometric Frobenius of C, then

$$\bar{L}(C,u) = \det(u.\mathbf{I} - q^{-1/2}F_C) = \det(u\,\mathbf{I} - \dot{m}(C)).$$

Then

$$|C(\mathbb{F}_q)| = q + 1 - q^{1/2}\boldsymbol{\tau}(C), \tag{4.1}$$

where $\boldsymbol{\tau}(C) = a_1(C)$, namely

$$\boldsymbol{\tau}(C) = q^{-1/2}\,\mathrm{Trace}\,F_C = 2\sum_{j=1}^{g}\cos\theta_j,$$

with $\theta(C) = (\theta_1, \dots, \theta_g)$.

Then *Katz-Sarnak theory* [10] models the behavior of the Weil polynomial of a random curve C of genus g over $\mathbb{F}_q$ by postulating that when q is large, the class $\dot{m}(C)$ behaves like a random conjugacy class in $\mathrm{Cl}\,G$, viewed as a probability space, endowed with the image $d\dot{m}$ of the mass one Haar measure. Here is an illustration of their results. Let $R(G)$ be the character ring of G (cf. the appendix) and

$$\mathscr{T}(G)^\circ = R(G) \otimes \mathbb{C} \simeq \mathbb{C}[2\cos\theta_1, \dots 2\cos\theta_g]^{\mathrm{sym}}$$

the algebra of *continuous representative central functions* on G, the isomorphism coming from Proposition A.1. This algebra is dense in $\mathscr{C}(G)^\circ$, hence, suitable for testing equidistribution on $\mathrm{Cl}\,G$. We use the following notation for the average of a complex function f defined over a finite set Z:

$$\oint_{\mathsf{Z}} f(z)dz = \frac{1}{|\mathsf{Z}|}\sum_{z\in\mathsf{Z}} f(z).$$

For every finite field k, we denote by $\mathsf{M}_g(k)$ the finite set of k-isomorphism classes of curves of genus g over k. The following theorem follows directly, if $g \geq 3$, from [10, Th. 10.7.15] (with a proof based on universal families of curves with a $3K$ structure), and from [10, Th. 10.8.2] if $g \leq 2$ (with a proof based on universal families of hyperelliptic curves).

Theorem 4.1 (Katz-Sarnak)**.** *Assume $g \geq 1$. If C runs over $\mathsf{M}_g(\mathbb{F}_q)$, the conjugacy classes $\dot{m}(C)$ become equidistributed in $\mathrm{Cl}\,G$ with respect to $d\dot{m}$ as $q \to \infty$. More precisely, if $\mathsf{F} \in \mathscr{T}(G)^\circ$, then*

$$\oint_{\mathsf{M}_g(\mathbb{F}_q)} \mathsf{F}(\dot{m}(C))dC = \int_{\mathrm{Cl}\,G} \mathsf{F}(m)\,dm + O\left(q^{-1/2}\right). \quad \square$$

Theorem 4.1 means that the counting measures

$$\mu_{g,q} = \frac{1}{|\mathsf{M}_g(\mathbb{F}_q)|}\sum_{C\in\mathsf{M}_g(\mathbb{F}_q)} \delta_{(\dot{m}(C))},$$

defined on $\mathrm{Cl}\,G$, converges to $d\dot{m}$ in the weak topology of measures when $q \to \infty$. Since

$$\mathsf{F} \circ h(\theta(C)) = \mathsf{F}(\dot{m}(C)),$$

this theorem means also that if C runs over $\mathsf{M}_g(\mathbb{F}_q)$, the vectors $\theta(C)$ become equidistributed in the fundamental alcove with respect to the Weyl measure when $q \to \infty$.

Remark 4.2. In the preceding theorem, and the above comments, one can substitute the set $\mathsf{A}_g(\mathbb{F}_q)$ to the set $\mathsf{M}_g(\mathbb{F}_q)$ [10, Th. 11.3.10]. This is an answer to the question raised in the beginning of this section.

As discussed above, the random variable $\boldsymbol{\tau}(C)$ rules the number of points on the set $\mathsf{M}_g(\mathbb{F}_q)$, and its law is the counting measure on the closed interval $[-2g, 2g]$:

$$\nu_{g,q} = \frac{1}{|\mathsf{M}_g(\mathbb{F}_q)|} \sum_{C \in \mathsf{M}_g(\mathbb{F}_q)} \delta_{(\boldsymbol{\tau}(C))} = \sum_{x=-2g}^{2g} f_{g,q}(x) \delta_{(x)},$$

where $\delta_{(x)}$ is the Dirac measure at x, with the probability mass function

$$f_{g,q}(x) = \frac{|\{C \in \mathsf{M}_g(\mathbb{F}_q) \mid (\boldsymbol{\tau}(C) = x\}|}{|\mathsf{M}_g(\mathbb{F}_q)|},$$

defined if $x \in [-2g, 2g]$ and $q^{1/2}x \in \mathbb{Z}$. We put now

$$\boldsymbol{\tau}(m) = \operatorname{Trace} m, \qquad \boldsymbol{\tau} \circ h(\theta) = 2 \sum_{j=1}^{g} \cos \theta_j,$$

for $m \in G$ and $\theta \in X_g$. We take $\mathsf{F}(m) = \boldsymbol{\tau}(m)$ in Theorem 4.1, and call $\mu_{\boldsymbol{\tau}}$ be the distribution of the central function $\boldsymbol{\tau}$ as defined at the end of Section 3. We obtain:

Corollary 4.3. *If $q \to \infty$, the distributions $\nu_{g,q}$ of the Frobenius traces converge to the distribution $\mu_{\boldsymbol{\tau}}$. More precisely, for any continuous function φ on $[-2g, 2g]$, we have*

$$\oint_{\mathsf{M}_g(\mathbb{F}_q)} \varphi(\boldsymbol{\tau}(C)) dC = \int_{-2g}^{2g} \varphi(x) \mu_{\boldsymbol{\tau}}(x) + O\left(q^{-1/2}\right),$$

and for every $x \in \mathbb{R}$, we have

$$\frac{|\{C \in \mathsf{M}_g(\mathbb{F}_q) \mid \boldsymbol{\tau}(C) \leq x\}|}{|\mathsf{M}_g(\mathbb{F}_q)|} = \int_{-\infty}^{x} f_{\boldsymbol{\tau}}(s) ds + O\left(q^{-1/2}\right). \quad \square$$

Lemma 4.4. *If $1 \leq n \leq 2g - 1$,*

$$\oint_{\mathsf{A}_g(\mathbb{F}_q)} a_n(A)\, dA = \varepsilon_n + O\left(q^{-1/2}\right),$$

where $\varepsilon_n = 1$ if n is even and $\varepsilon_n = 0$ if n is odd.

Proof. As Equation (A.4) in the appendix, let

$$\boldsymbol{\tau}_n(m) = \operatorname{Trace}(\wedge^n m),$$

in such a way that $\boldsymbol{\tau}_1 = \boldsymbol{\tau}$. By equality (A.5), we have

$$a_n(A) = \boldsymbol{\tau}_n \circ h(\theta(A)).$$

Since $\boldsymbol{\tau}_n \in \mathscr{T}(G)^\circ$, we have, by Remark 4.2,

$$\oint_{\mathsf{A}_g(\mathbb{F}_q)} \boldsymbol{\tau}_n \circ h(\theta(A))\, dA = \int_G \boldsymbol{\tau}_n(m)\, dm + O\left(q^{-1/2}\right)$$

but Lemma A.6 implies that the multiplicity of the character $\boldsymbol{\tau}_0$ of the unit representation $\mathbf{1}$ is equal to ε_n, hence,

$$\int_G \boldsymbol{\tau}_n(m)\, dm = \varepsilon_n. \quad \square$$

Corollary 4.5. *Let $u \in \mathbb{C}$ and $q \to \infty$.*

(i) *If $|u| < q^{1/2}$, then*

$$\oint_{\mathsf{A}_g(\mathbb{F}_q)} L(A,u)\, dA = \frac{u^{2g+2} - q^{g+1}}{u^2 - q} + O\left(q^{g-\frac{1}{2}}\right).$$

(ii) *We have*

$$\oint_{\mathsf{A}_g(\mathbb{F}_q)} |A(\mathbb{F}_q)|\, dA = q^g + O\left(q^{g-1}\right).$$

(iii) *We have*

$$\oint_{\mathsf{M}_g(\mathbb{F}_q)} |C(\mathbb{F}_q)|\, dC = q + O(1).$$

The implied constants depend only on g.

Proof. We have

$$L(A,u) = q^g \bar{L}\left(A, q^{-1/2}u\right) = \sum_{n=0}^{2g} (-1)^n a_n(A) q^{(2g-n)/2} u^n,$$

with $a_0 = 1$ and $a_{2g-n} = a_n$ for $0 \le n \le g$. From Lemma 4.4, we get

$$q^{(2g-n)/2} u^n \oint_{\mathsf{F}_g(\mathbb{F}_q)} a_n(A)\, dA = \varepsilon_n q^{(2g-n)/2} u^n + u^n O\left(q^{(2g-n-1)/2}\right)$$

for $1 \le n \le 2g-1$, and there is no second term in the right hand side if $n = 0$ and $n = 2g$. Now, if $|u| < q^{1/2}$,

$$\sum_{n=0}^{2g} \varepsilon_n q^{(2g-n)/2} u^n = \frac{u^{2g+2} - q^{g+1}}{u^2 - q},$$

and the absolute value of the difference between this expression and

$$\oint_{\mathsf{A}_g(\mathbb{F}_q)} L(A,u)\, dA$$

is bounded by

$$B \sum_{n=1}^{2g-1} |u|^n q^{(2g-n-1)/2},$$

with B depending only on g. If $|u| \le q^{1/2}$, then $|u|^n q^{(2g-n-1)/2} \le q^{(2g-1)/2}$, and this proves (i). If $|u| \le 1$, then $|u|^n q^{(2g-n-1)/2} \le q^{g-1}$, hence,

$$\oint_{\mathsf{A}_g(\mathbb{F}_q)} L(A,u)\, dA = q^g + O\left(q^{g-1}\right).$$

and this proves (ii), since $|A(\mathbb{F}_q)| = L(A,1)$. Since Lemma 4.4 holds by substituting M_g to A_g, (iii) is a consequence of this lemma applied to $a_1(C)$, and of formula (4.1). $\square$

With Corollary 4.5(i), it appears as though the Frobenius angles were close in the mean to the vertices of the regular polygon with $(2g+2)$ vertices, inscribed in the circle of radius $q^{1/2}$, the points $\pm q^{1/2}$ being excluded.

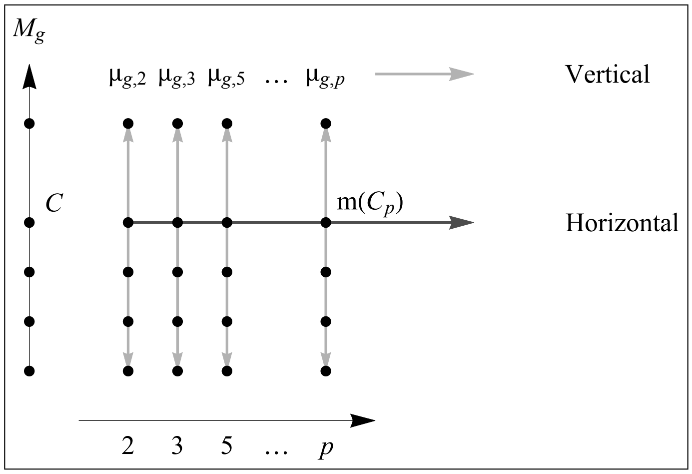

Figure 1. Horizontal versus vertical distribution.

Another approach on the limiting equidistribution of Frobenius angles is the *generalized Sato-Tate conjecture*, see [17] for a comprehensive description. Let C be an absolutely irreducible nonsingular projective curve of genus g over $\mathbb{Q}$, and S a finite subset of prime numbers such that the reduction $C_p = C_{\mathbb{F}_p}$ over $\mathbb{F}_p$ is good if $p \notin S$. Then one says that the group $\mathbf{USp}_{2g}$ *arises as the Sato-Tate group* of C if the conjugacy classes $\dot{m}(C_p)$ are equidistributed with respect to the Weyl measure of G when $p \to \infty$. In other words, this means that if $\mathsf{F} \in \mathscr{C}(G)^\circ$, then

$$\lim_{n\to\infty} \oint_{P_S(n)} \mathsf{F}(\dot{m}(C_p))\, dp = \int_G \mathsf{F}(m)\, dm,$$

where $P_S(n) = \{p \in P \setminus S \mid p \leq n\}$. The case $g = 1$ is the Sato-Tate original conjecture, now a theorem. Here is an example of what one expects [12] :

Conjecture 4.6 (Kedlaya-Sutherland)**.** *If* $\mathrm{End}_{\mathbb{C}}(\mathrm{Jac}\, C) = \mathbb{Z}$*, and if* g *is odd, or* $g = 2$*, or* $g = 6$*, then the group* $\mathbf{USp}_{2g}$ *arises as the Sato-Tate group of* C*.*

The two preceding types of equidistribution are symbolically shown in Figure 1. The sequence of prime numbers are on the horinzontal axis, and the vertical axis symbolizes the space of curves. The Katz-Sarnak approach is figured as a (horizontal) limit of vertical averages μ_p over vertical lines, and the Sato-Tate approach is a mean performed on horizontal lines.

5. Expressions of the law of the trace in genus 2

Assume now $g = 2$. Our purpose is to express the density of the distribution of the trace function $\boldsymbol{\tau}$ on $\mathbf{USp}_4$ with the help of special functions. In order to do this, the first step is to compute the characteristic function. The density of the Weyl measure on X_2 is

$$\delta_2(\theta_1, \theta_2) = \left(\frac{2}{\pi^2}\right) \sin^2\theta_1 \sin^2\theta_2 (2\cos\theta_2 - 2\cos\theta_1)^2.$$

The fundamental alcove is

$$A_2 = \{(\theta_1, \theta_2) \in X_2 \mid 0 < \theta_2 < \theta_1 < \pi\}.$$

The maximum of δ_2 in A_2 is attained at the point

$$\theta_m = (\alpha_m, \pi - \alpha_m), \text{ where } \tan\frac{\alpha_m}{2} = \sqrt{2+\sqrt{3}}, \quad \delta(\theta_m) = \frac{128}{27\pi^2}.$$

We have $\boldsymbol{\tau} \circ h(\theta_1, \theta_2) = 2\cos\theta_1 + 2\cos\theta_2$, and the characteristic function of $\boldsymbol{\tau}$ is

$$\widehat{f}_{\boldsymbol{\tau}}(t) = \int_{X_2} e^{2it(\cos\theta_1 + \cos\theta_2)} \delta_2(\theta_1, \theta_2) d\theta_1 d\theta_2.$$

Proposition 5.1. *For every $t \in \mathbb{R}$, we have*

$$\widehat{f}_{\boldsymbol{\tau}}(t) = \frac{4J_1(2t)^2}{t^2} - \frac{6J_1(2t)J_2(2t)}{t^3} + \frac{4J_2(2t)^2}{t^2}.$$

Here, J_1 and J_2 are *Bessel functions.*

Proof. Let

$$\begin{array}{rclcl} V_a(x) & = & 2^5\cos^2 x \sin^2 x & = & 8\sin^2(2x) \\ V_b(x) & = & 2^5\cos^2 x \sin^2 x \cos 2x & = & 4\sin 2x \cos 4x \\ V_c(x) & = & 2^5\cos^2 x \sin^2 x \cos^2 2x & = & 2\sin^2 4x \end{array}$$

Then

$$32\pi^2\delta(x, y) = V_c(x)V_a(y) + V_a(x)V_c(y) - 2V_b(x)V_b(y).$$

and

$$\widehat{F}(t) = 2\widehat{V}_a(t)\widehat{V}_c(t) - 2\widehat{V}_b(t)^2.$$

But

$$\widehat{V}_a(t) = \frac{\sqrt{2}}{t} J_1(2t), \quad \widehat{V}_b(t) = \frac{i\sqrt{2}}{t} J_2(2t), \quad \widehat{V}_c(t) = \frac{\sqrt{2}}{t} J_1(2t) - \frac{3}{\sqrt{2}t^2} J_2(2t),$$

and the result follows. $\square$

We now compute the moments $M_n(\boldsymbol{\tau})$ of $\boldsymbol{\tau}$. By Proposition 5.1, the characteristic function can be expressed by a *generalized hypergeometric series* [9, §9.14, p. 1010]:

$$\widehat{f}_{\boldsymbol{\tau}}(t) = {}_1F_2\left(\frac{3}{2}; 3, 4; -4t^2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{3}{2})_n}{(3)_n(4)_n} 2^{2n} \frac{t^{2n}}{n!},$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer's symbol. It then follows from (3.6) that the odd moments are equal to zero. Since

$$\frac{(\frac{3}{2})_n}{(3)_n(4)_n} = \frac{24}{\sqrt{\pi}} \frac{(n+\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+3)\Gamma(n+4)}$$

and [9, p. 897]

$$\Gamma\left(n + \frac{1}{2}\right) = \sqrt{\pi}\, 2^{-2n} \frac{(2n)!}{n!},$$

we obtain

$$M_{2n}(\boldsymbol{\tau}) = \frac{6.(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!} \quad \text{for } n \geq 0.$$

One finds as expected Mihailovs' formula, in accordance with [12, §4.1], which includes another formula for $\widehat{f_{\boldsymbol{\tau}}}(t)$, and also [17, p. 126].

In what follows, four different but equivalent expressions for the distribution of τ are given.

5.1. **Hypergeometric series.** An expression of the density $f_{\boldsymbol{\tau}}$ of the distribution of $\boldsymbol{\tau}$ is the following. Recall that *Gauss' hypergeometric series*

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

converges if $|z| < 1$ [9, §9.1.0, p. 1005].

Theorem 5.2. *If $|x| < 4$, we have*

$$f_{\boldsymbol{\tau}}(x) = \frac{1}{4\pi}\left(1 - \frac{x^2}{16}\right)^4 {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 5; 1 - \frac{x^2}{16}\right).$$

This theorem immediately follows from the following lemma.

Lemma 5.3. *If $|x| < 4$, we have*

$$f_{\boldsymbol{\tau}}(x) = \frac{64}{5\pi^2} m(x)^4 I(m(x)), \quad \textit{where} \quad m(x) = 1 - \frac{x^2}{16},$$

and

$$I(m) = \int_0^1 t^2 \left(\frac{1-t^2}{1-mt^2}\right)^{\frac{5}{2}} dt.$$

Moreover

$$I(m) = \frac{5\pi}{256} \, {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 5; m\right).$$

Proof. We use a *formula of Schläfli*, see [18, Eq. 1, p. 150]. If μ and ν are real numbers, then

$$J_\mu(t) J_\nu(t) = \frac{2}{\pi} \int_0^{\pi/2} J_{\mu+\nu}(2t\cos\varphi)\cos(\mu-\nu)\varphi \, d\varphi \quad (\mu+\nu > -1).$$

As particular cases of this formula, we get

$$J_1(t)^2 = \frac{2}{\pi}\int_0^4 J_2\left(\frac{ut}{2}\right) \frac{du}{\sqrt{16-u^2}}$$

$$J_1(t)J_2(t) = \frac{2}{\pi}\int_0^4 J_3\left(\frac{ut}{2}\right) \frac{u}{4} \frac{du}{\sqrt{16-u^2}},$$

$$J_2(t)^2 = \frac{2}{\pi}\int_0^4 J_4\left(\frac{ut}{2}\right) \frac{du}{\sqrt{16-u^2}}.$$

By transferring these equalities in Proposition 5.1, we obtain

$$\begin{aligned}\widehat{f_{\boldsymbol{\tau}}}(t) &= \frac{4}{t^2} J_1(2t)^2 - \frac{6}{t^3} J_1(2t) J_2(2t) + \frac{4}{t^2} J_2(2t)^2 \\ &= \frac{2}{\pi}\int_0^4 \left[\frac{4}{t^2} J_2(ut) - \frac{3u}{2t^3} J_3(ut) + \frac{4}{t^2} J_4(ut)\right] \frac{du}{\sqrt{16-u^2}}.\end{aligned}$$

and since

$$f_{\boldsymbol{\tau}}(x) = \frac{1}{\pi}\int_0^\infty \widehat{f_{\boldsymbol{\tau}}}(t)\cos tx\,dt,$$

we have

$$f_{\boldsymbol{\tau}}(x) = \frac{2}{\pi^2}\int_0^4 \frac{du}{\sqrt{16-u^2}}\int_0^\infty \left[\frac{4}{t^2}J_2(ut) - \frac{3u}{2t^3}J_3(ut) + \frac{4}{t^2}J_4(ut)\right]\cos tx\,dt. \tag{5.1}$$

We use now a *formula of Gegenbauer* on the cosine transform, see [15, p. 409] and [18, Eq. 3, p. 50]. Assume $\operatorname{Re}\nu > -1/2$, $u > 0$ and let n be an integer ≥ 0. If $0 < x < u$, then

$$\int_0^\infty t^{-\nu}J_{\nu+2n}(ut)\cos tx\,dt = (-1)^n 2^{\nu-1}u^{-\nu}\frac{\Gamma(\nu)}{\Gamma(2\nu+n)}(u^2-x^2)^{\nu-1/2}C_{2n}^{\nu}\left(\frac{x}{u}\right),$$

where $C_n^\nu(x)$ is the *Gegenbauer polynomial.* If $u < x < \infty$, this integral is equal to 0. From Gegenbauer's formula we deduce that if $0 < x < u$, then

$$\int_0^\infty t^{-2}J_2(ut)\cos tx\,dt = \frac{1}{3}\frac{(u^2-x^2)^{3/2}}{u^2},$$

$$\int_0^\infty t^{-3}J_3(ut)\cos tx\,dt = \frac{1}{15}\frac{(u^2-x^2)^{5/2}}{u^3},$$

$$\int_0^\infty t^{-2}J_4(ut)\cos tx\,dt = -\frac{1}{30}\frac{(u^2-x^2)^{3/2}}{u^2}\left(\frac{12x^2}{u^2}-2\right),$$

since $C_2^2(x) = 12x^2 - 2$. Transferring these relations in (5.1), we get

$$\begin{aligned} 5\pi^2 f_{\boldsymbol{\tau}}(x) = 16\int_x^4 &\frac{(u^2-x^2)^{3/2}}{u^2}\frac{du}{\sqrt{16-u^2}} \\ &- \int_x^4 \frac{(u^2-x^2)^{5/2}}{u^2}\frac{du}{\sqrt{16-u^2}} - 16x^2\int_x^4 \frac{(u^2-x^2)^{3/2}}{u^4}\frac{du}{\sqrt{16-u^2}}, \end{aligned}$$

and this leads to

$$f_{\boldsymbol{\tau}}(x) = \frac{1}{5\pi^2}\int_x^4 \frac{(u^2-x^2)^{5/2}}{u^4}\sqrt{16-u^2}\,du.$$

By the change of variables

$$u = 4\sqrt{1-m(x)t^2}, \quad \text{where} \quad m(x) = 1 - \frac{x^2}{16}.$$

we obtain

$$f_{\boldsymbol{\tau}}(x) = \frac{64\,m(x)^4}{5\pi^2}\int_0^1 t^2\left(\frac{1-t^2}{1-m(x)t^2}\right)^{\frac{5}{2}}dt,$$

which is the first result. Euler's integral representation of the hypergeometric series is

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt$$

if $\operatorname{Re} z > 0$, and $\operatorname{Re} c > \operatorname{Re} b > 0$. From this we deduce, with the change of variables $t = u^2$, that

$$I(m) = \frac{5\pi}{256}\,{}_2F_1\left(\frac{3}{2},\frac{5}{2};5;m\right),$$

which is the second result. $\square$

5.2. **Legendre function.** Another expression of $f_{\boldsymbol{\tau}}$ is given by the *associated Legendre function of the first kind* $\mathcal{P}_b^a(z)$, defined in the half-plane $\operatorname{Re} z > 1$. If a is not an integer ≥ 1, and if $b > 3/2$, this function is defined by [9, Eq. 8.702, p. 959] :

$$\mathcal{P}_b^a(z) = \frac{1}{\Gamma(1-a)} \left(\frac{z+1}{z-1}\right)^{\frac{a}{2}} {}_2F_1\left(-b, b+1; 1-a; \frac{1-z}{2}\right).$$

If $a = m$ is an integer and if $z > 1$ is real, then [9, Eq. 8.711.2, p. 960] :

$$\mathcal{P}_b^m(z) = \frac{(b+1)_a}{\pi} \int_0^\pi \left(z + \sqrt{z^2-1}\cos\varphi\right)^b \cos m\varphi \, d\varphi.$$

If $a = 0$, this is the *Laplace integral.*

Theorem 5.4. *If $|x| < 4$, then*

$$f_{\boldsymbol{\tau}}(x) = -\frac{64}{15\pi} \sqrt{|x|} \left(1 - \frac{x^2}{16}\right)^2 \mathcal{P}_{\frac{1}{2}}^2 \left(\frac{x^2+16}{4x}\right).$$

Proof. By Theorem 5.2, we have

$$F(x) = \frac{1}{4\pi} m(x)^4 \, {}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 5; m(x)\right).$$

But [15, p. 51]

$${}_2F_1\left(\frac{3}{2}, \frac{5}{2}; 5; z\right) = (1-z)^{-3/4} {}_2F_1\left(\frac{3}{2}, \frac{7}{2}; 3; -\frac{(1-\sqrt{1-z})^2}{4\sqrt{1-z}}\right)$$

and [15, p. 47]

$${}_2F_1\left(\frac{3}{2}, \frac{7}{2}; 3; z\right) = (1-z)^{-2} {}_2F_1\left(-\frac{1}{2}, \frac{3}{2}; 3; z\right).$$

On the other hand, if $z = m(x)$, then

$$-\frac{(1-\sqrt{1-z})^2}{4\sqrt{1-z}} = -\frac{(x-4)^2}{16x}.$$

By the definition of Legendre functions,

$$\mathcal{P}_{\frac{1}{2}}^{-2}\left(\frac{1}{2}\left(\frac{x}{4} + \frac{4}{x}\right)\right) = \left(\frac{x-4}{x+4}\right)^4 \, {}_2F_1(-\frac{1}{2}, \frac{3}{2}; 3; -\frac{(x-4)^2}{16x}),$$

and this implies

$$f_{\boldsymbol{\tau}}(x) = \frac{4}{\pi} \sqrt{x} \left(1 - \frac{x^2}{16}\right)^2 \mathcal{P}_{\frac{1}{2}}^{-2}\left(\frac{1}{2}\left(\frac{x}{4} + \frac{4}{x}\right)\right).$$

Since

$$\mathcal{P}_b^m(z) = \frac{\Gamma(b+m+1)}{\Gamma(b-m+1)} \mathcal{P}_b^{-m}(z)$$

if $m \in \mathbb{Z}$, we obtain the required expression. □

Since ${}_2F_1(a, b; c; 0) = 1$, we deduce from Theorem 5.2 that

$$f_{\boldsymbol{\tau}}(x) = \frac{1}{4\pi}\left(1 - \frac{x^2}{16}\right)^4 + O(x-4)^5$$

and hence, in accordance with [17, p. 126]:

Corollary 5.5. *If* $|x| = 4 - \varepsilon$*, with* $\varepsilon \to 0$ *and* $\varepsilon > 0$*, then*

$$f_{\boldsymbol{\tau}}(x) = \frac{\varepsilon^4}{64\pi} + O(\varepsilon^5). \qquad \square$$

Since

$$\lim_{x \to 0} \sqrt{x}\, \mathcal{P}^2_{\frac{1}{2}} \left(\frac{1}{2} (\frac{x}{4} + \frac{4}{x}) \right) = -\frac{1}{\pi} \, ,$$

we deduce from Proposition 5.4 that the maximum of $f_{\boldsymbol{\tau}}$ is reached for $x = 0$, and

$$f_{\boldsymbol{\tau}}(0) = \frac{64}{15\pi^2} = 0.432\ldots$$

The graph of $f_{\boldsymbol{\tau}}$ is given in Figure 2 ; we recover the curve drawn in [12, p. 124].

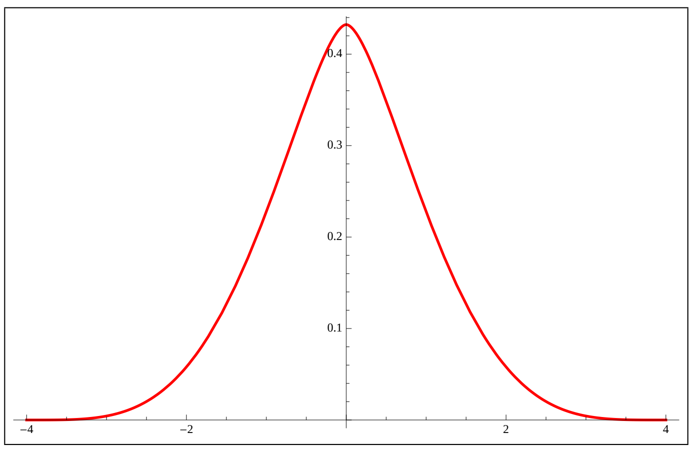

FIGURE 2. Density of the distribution of $\boldsymbol{\tau}$, case $g = 2$.

5.3. **Elliptic integrals.** Another expression of $f_{\boldsymbol{\tau}}$ is given by *Legendre elliptic integrals.* Let

$$K(m) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}}, \qquad E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \varphi}\, d\varphi,$$

be the Legendre elliptic integrals of first and second kind, respectively. The implementation of $f_{\boldsymbol{\tau}}$ in the *Maple* software gives:

Corollary 5.6. *If* $|x| < 4$*, then*

$$\frac{15}{64}\pi^2 f_{\boldsymbol{\tau}}(x) = (m^2 - 16m + 16)E(m) - 8(m^2 - 3m + 2)K(m),$$

where $m = 1 - (x^2/16)$. $\square$

The mention of the existence of such a formula is made in [7].

5.4. **Meijer G-functions.** Another expression of $f_{\boldsymbol{\tau}}$ is given by *Meijer G-functions* [9, §9.3, p. 1032]. They are defined as follows : take z in $\mathbb{C}$ with $0 < |z| < 1$ and m, n, p, q in $\mathbb{N}$. Then

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) = \frac{1}{2i\pi}\int_C \frac{\prod_{k=1}^m \Gamma(s+b_k)}{\prod_{k=n+1}^p \Gamma(s+a_k)} \cdot \frac{\prod_{k=1}^n \Gamma(-s-a_k+1)}{\prod_{k=m+1}^q \Gamma(-s-b_k+1)}\, z^{-s} ds$$

Here, $a_1,\dots,a_p,b_1,\dots,b_q$ are *a priori* in $\mathbb{C}$, and C is a suitable Mellin-Barnes contour.

Corollary 5.7. *If $|x| < 4$,*

$$f_{\boldsymbol{\tau}}(x) = \frac{6}{\pi} G\left(\frac{x^2}{16}\right), \quad \textit{with} \quad G(z) = G^{2,0}_{2,2}\left(z \,\middle|\, \begin{matrix} \frac{5}{2}, & \frac{7}{2} \\ 0, & 1 \end{matrix}\right).$$

We have

$$G(z) = \frac{1}{2i\pi}\int_{\operatorname{Re} s = c} \frac{\Gamma(s)\Gamma(s+1)}{\Gamma\left(s+\frac{5}{2}\right)\Gamma\left(s+\frac{7}{2}\right)}\, z^{-s} ds,$$

with $0 < c < \frac{1}{2}$.

Proof. If $|z| < 1$, then [19, 07.34.03.0653.01]:

$$G^{2,0}_{2,2}\left(z \,\middle|\, \begin{matrix} a, c \\ b, -a+b+c \end{matrix}\right) = \frac{\sqrt{\pi}}{\Gamma(a-b)}(1-z)^{a-b-\frac{1}{2}} z^{\frac{1}{4}(-2a+2c-1)+b} \mathfrak{P}^{-a+b+\frac{1}{2}}_{-a+c-\frac{1}{2}}\left(\frac{z+1}{2\sqrt{z}}\right)$$

and the left hand side is equal to zero if $|z| > 1$. Hence, if $|z| < 1$,

$$G^{2,0}_{2,2}\left(z \,\middle|\, \begin{matrix} \frac{5}{2}, & \frac{7}{2} \\ 0, & 1 \end{matrix}\right) = \frac{4}{3}(1-z)^2 z^{1/4}\, \mathfrak{P}^{-2}_{1/2}\left(\frac{z+1}{2\sqrt{z}}\right),$$

and we apply Theorem 5.4. □

Corollary 5.8. *If $|x| < 4$, then the repartition function of $\boldsymbol{\tau}$ is*

$$\Phi_{\boldsymbol{\tau}}(x) = \frac{3x}{\pi} G\left(\frac{x^2}{16}\right) + \frac{1}{2},$$

with

$$G(z) = G^{2,1}_{3,3}\left(z \,\middle|\, \begin{matrix} \frac{1}{2}, & \frac{5}{2}, & \frac{7}{2} \\ 0, & 1, & -\frac{1}{2} \end{matrix}\right).$$

Proof. According to [19, 07.34.21.0003.01], we have

$$\int z^{\alpha-1} G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}\right) dz = z^{\alpha} G^{m,n+1}_{p+1,q+1}\left(z \,\middle|\, \begin{matrix} 1-\alpha, a_1,\dots,a_p \\ b_1,\dots,b_q,-\alpha \end{matrix}\right).$$

By Corollary 5.7, a primitive of $f_{\boldsymbol{\tau}}$ is

$$\Phi_0(x) = \frac{6}{\pi}\int G^{2,0}_{2,2}\left(\frac{x^2}{16} \,\middle|\, \begin{matrix} \frac{5}{2}, & \frac{7}{2} \\ 0, & 1 \end{matrix}\right) = \frac{3x}{\pi} G^{2,1}_{3,3}\left(\frac{x^2}{16} \,\middle|\, \begin{matrix} \frac{1}{2}, & \frac{5}{2}, & \frac{7}{2} \\ 0, & 1, & -\frac{1}{2} \end{matrix}\right),$$

and $\Phi_0(-4) = -1/2$. □

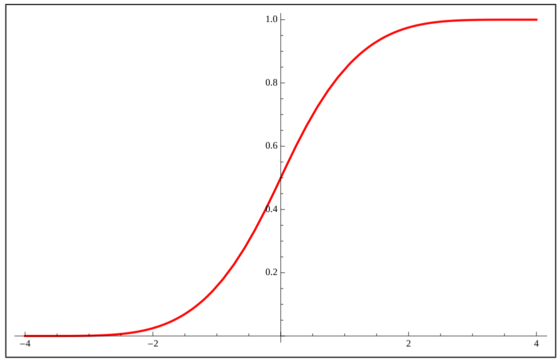

FIGURE 3. Repartition function of $\boldsymbol{\tau}$

5.5. **The trace in $\mathbf{SU}_2 \times \mathbf{SU}_2$.** In order to present a comparison with the above results, we give here without proof the distribution of the trace of a compact semi-simple subgroup of rank 2 of $\mathbf{USp}_4$, namely, the group $\mathbf{SU}_2 \times \mathbf{SU}_2$. If

$$x = (x_1, x_2) \quad \text{and} \quad x_i = \begin{pmatrix} a_i & -\bar{b}_i \\ b_i & \bar{a}_i \end{pmatrix} \in \mathbf{SU}_2, \quad |a_i|^2 + |b_i|^2 = 1, \quad i = 1, 2,$$

the map

$$\pi(x) = \begin{pmatrix} a_1 & 0 & -\bar{b}_1 & 0 \\ 0 & a_2 & 0 & -\bar{b}_2 \\ b_1 & 0 & \bar{a}_1 & 0 \\ 0 & b_2 & 0 & \bar{a}_2 \end{pmatrix}$$

defines an embedding

$$\pi : \mathbf{SU}_2 \times \mathbf{SU}_2 \longrightarrow \mathbf{USp}_4$$

whose image contains the maximal torus T of $\mathbf{USp}_4$. We put

$$\boldsymbol{\rho}(x) = \operatorname{Trace} \pi(x).$$

The characteristic function of $\boldsymbol{\rho}$ is the square of the characteristic function of the distribution of the trace of $\mathbf{SU}_2$:

$$\widehat{f_{\boldsymbol{\rho}}}(t) = \frac{J_1(2t)^2}{t^2}.$$

The even moments are equal to zero, and the odd moments are

$$M_{2n}(\boldsymbol{\rho}) = C_n C_{n+1} = \frac{2(2n)!(2n+1)!}{(n!)^2(n+1)!(n+2)!}\,.$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the nth *Catalan number*. One finds the sequence

$$1, 0, 2, 0, 10, 0, 70, 0, 588, 0, 5544\ldots$$

in accordance with the sequence A005568 in the OEIS [16].

Theorem 5.9. *If $|x| < 4$, the density of the distribution of $\boldsymbol{\rho}$ is*

$$f_{\boldsymbol{\rho}}(x) = \frac{1}{2\pi}\left(1 - \frac{x^2}{16}\right)^2 {}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 3; 1 - \frac{x^2}{16}\right). \qquad \square$$

Corollary 5.10. *If $|x| = 4 - \varepsilon$, with $\varepsilon \to 0$ and $\varepsilon > 0$, then*

$$f_{\boldsymbol{\rho}}(x) = \frac{\varepsilon^2}{8\pi} - \frac{\varepsilon^3}{64\pi} - \frac{\varepsilon^4}{4096\pi} + O(\varepsilon^5). \qquad \square$$

The maximum of $f_{\boldsymbol{\rho}}$ is reached for $x = 0$, and

$$f_{\boldsymbol{\rho}}(0) = \frac{8}{3\pi^2} = 0.270\ldots$$

The graph of $f_{\boldsymbol{\rho}}$ is given in Figure 4.

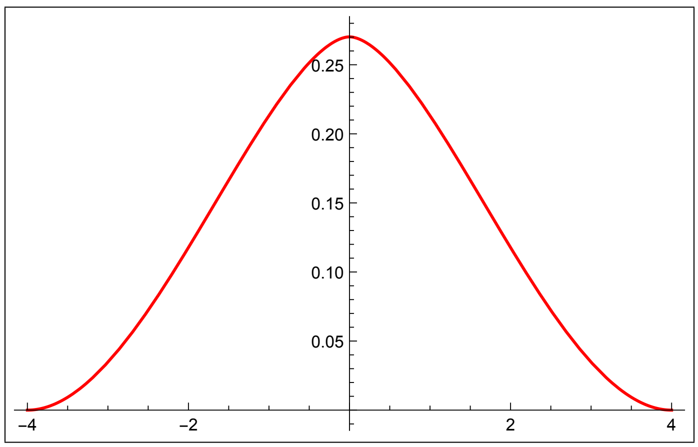

FIGURE 4. Density of the distribution of $\boldsymbol{\rho}$.

6. The Viète map and its image

Another approach of the distribution of the trace is realized by an algebraic form of Weyl's integration formula, using symmetric polynomials. This comes from a general program developed by Kohel [13], formerly outlined by DiPippo and Howe in [6]. If $t = (t_1, \ldots, t_g) \in \mathbb{C}^g$, consider a monic polynomial

$$h_t(u) = (u - t_1)\ldots(u - t_g) = u^g - s_1(t)u^{g-1} + \cdots + (-1)^g s_g(t) \tag{6.1}$$

in $\mathbb{C}[u]$. Here

$$s_n(t) = \sum_{i_1 < \cdots < i_n} t_{i_1} \ldots t_{i_k}$$

is the *elementary symmetric polynomial* of degree n in g variables. The discriminant of h_t is

$$\operatorname{disc} h_t = D_0(t) = \prod_{j<k} (t_k - t_j)^2. \tag{6.2}$$

The *Viète map* $\mathsf{s} : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ is the surjective polynomial mapping

$$\mathsf{s}(t_1, \ldots, t_g) = (s_1(t), \ldots, s_g(t)),$$

where $t = (t_1, \dots, t_g)$, inducing a bijection

$$\mathbb{C}^g/\mathfrak{S}_g \xrightarrow{\sim} \mathbb{C}^g$$

which is a homeomorphism, because the map between the corresponding projective spaces is a continuous bijection between compact spaces. Hence, the Viète map is open and proper. We denote by

$$\Pi_g = \mathsf{s}(\mathbb{R}^g)$$

the closed subset which is the image of the Viète map. Hence, $(s_1, \dots, s_g) \in \Pi_g$ if and only $h_t(u)$ has only real roots. The induced map

$$\mathbb{R}^g/\mathfrak{S}_g \xrightarrow{\sim} \Pi_g$$

is a homeomorphism. The *fundamental chamber* of $\mathbb{R}^g$ related to $\mathfrak{S}_g$ is

$$C_g = \{t \in \mathbb{R}^g \mid t_1 < t_2 < \dots < t_g\}$$

and $\bar{C}_g$ is a fundamental domain for $\mathfrak{S}_g$ in $\mathbb{R}^g$. We are going to show that s is a local diffeomorphism at the points of an open dense subset of $\mathbb{R}^g$. For this purpose, we calculate $J(\mathsf{s})$, where $J(\mathsf{f})$ denotes the jacobian matrix of a polynomial map $\mathsf{f} : \mathbb{C}^g \longrightarrow \mathbb{C}^g$. Recall that the power sums

$$p_n(t) = t_1^n + \dots + t_g^n \quad (n \geq 1)$$

can be expressed in terms of elementary symmetric polynomials. Precisely, from Newton's relations

$$p_n = \sum_{j=1}^{n-1} (-1)^{j-1} s_j p_{n-j} + (-1)^{n-1} n s_n \quad (n \geq 1),$$

we obtain [14, p. 28] :

$$p_n = \begin{vmatrix} s_1 & 1 & 0 & \dots & 0 \\ 2s_2 & s_1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ ns_n & s_{n-1} & s_{n-2} & \dots & s_1 \end{vmatrix}.$$

This is related to a more suitable expression [5, p. 72], [2, Ch. IV, § 6, Ex. 6], obtained by Albert Girard [8] in 1629, and sometimes attributed to Waring (1762):

Proposition 6.1 (Girard's formula)**.** *If $1 \leq n \leq g$ and $s = (s_1, \dots, s_g)$, let*

$$v_n(s) = n \sum_{b \in \mathcal{P}_n} \frac{(b_1 + b_2 + \dots + b_g - 1)!}{b_1! \dots b_g!} \, u_1^{b_1} \dots u_g^{b_g},$$

where $u_n = (-1)^{n-1} s_n$ for $1 \leq n \leq g$, and the summation being extended to

$$\mathcal{P}_n = \{b = (b_1, \dots, b_g) \in \mathbb{N}^g \mid b_1 + 2b_2 + \dots + g b_g = n\}.$$

Then

$$p_n = v_n \circ \mathsf{s}. \quad \square$$

The map $\varphi \mapsto \varphi \circ \mathsf{s}$ defines an isomorphism

$$\mathsf{s}^* : \mathbb{Z}[s_1, \dots, s_g] \xrightarrow{\sim} \mathbb{Z}[t_1, \dots, t_g]^{\mathrm{sym}}.$$

Since $D_0 \in \mathbb{Z}[t_1, \dots, t_g]^{\mathrm{sym}}$, there is a polynomial $d_0 \in \mathbb{Z}[s_1, \dots, s_g]$ such that

$$d_0(\mathsf{s}(t)) = D_0(t) = \prod_{j<k} (t_k - t_j)^2. \tag{6.3}$$

Let

$$U_g = \{t \in \mathbb{R}^g \mid D_0(t) \neq 0\}\,, \quad \Pi_g^\circ = \{s \in \mathbb{R}^g \mid d_0(s) \neq 0\}\,.$$

Then $\Pi_g^\circ = \mathsf{s}(U_g)$, and Π_g° is a dense open set of Π_g. The roots of the polynomial $h_t \in \mathbb{R}[u]$ given by (6.1) are real and simple if and only if $\mathsf{s}(t) \in \Pi_g^\circ$.

Proposition 6.2. *With the preceding notation:*

(i) *If $t \in \mathbb{R}^g$, then*

$$|\det J(\mathsf{s})(t)| = \sqrt{D_0(t)} = \prod_{j<k} |t_k - t_j|.$$

(ii) *The map s is a local diffeomorphism at every point of U_g.*

(iii) *The map s is a diffeomorphism from the fundamental chamber C_g to Π_g°.*

Proof. Define two polynomial maps from $\mathbb{C}^g$ to $\mathbb{C}^g$:

$$\mathsf{p}(t) = (p_1(t), \dots, p_g(t)) \quad \text{and} \quad \mathsf{v}(s) = (v_1(s), \dots, v_g(s)).$$

Then $\mathsf{p} = \mathsf{v} \circ \mathsf{s}$ by Girard's formula 6.1. If $1 \leq n \leq g$, then

$$v_n(s) = (-1)^{n+1} n s_n + v_n'(s),$$

where $v_n'(s)$ depends only of $s_1, \dots, s_{n-1}$. This implies that $J(\mathsf{v})$ is lower triangular, with n-th diagonal term equal to $(-1)^{n+1}n$. Hence,

$$\det J(\mathsf{v}) = (-1)^{[g/2]}\, g!$$

On the other hand,

$$J(\mathsf{p}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ kt_1^{k-1} & kt_2^{k-1} & \dots & kt_n^{k-1} \\ \dots & \dots & \dots & \dots \\ gt_1^{g-1} & gt_2^{g-1} & \dots & gt_n^{g-1} \end{pmatrix}.$$

Then $J(\mathsf{p}) = D.V(t)$, where D is the diagonal matrix $\mathrm{diag}(1, 2, \dots, g)$, and

$$V(t) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ t_1 & t_2 & \dots & t_n \\ \dots & \dots & \dots & \dots \\ t_1^{g-1} & t_2^{g-1} & \dots & t_n^{g-1} \end{pmatrix}$$

is the Vandermonde matrix. Hence,

$$\det J(\mathsf{p}) = g! \det V(t) = g! \prod_{j<k} (t_k - t_j),$$

and since $J(\mathsf{p}) = J(\mathsf{v}).J(\mathsf{s})$, we get (i), which implies (ii). Then (iii) comes from the fact that s is injective on the open subset C_g of U_g. $\square$

The *bezoutian* of h_t is the matrix

$$B(t) = V(t).{}^tV(t) = \begin{pmatrix} p_0 & p_1 & \dots & p_{g-1} \\ p_1 & p_2 & \dots & p_g \\ \dots & \dots & \dots & \dots \\ p_{g-1} & p_g & \dots & p_{2g-2} \end{pmatrix} \in \mathbf{M}_g(\mathbb{R}),$$

in such a way that $\det B(t) = D_0(t)$.

Lemma 6.3. *Let $h_t \in \mathbb{R}[u]$. The following conditions are equivalent:*

(i) *The roots of* h_t *are real and simple, i.e.* $\mathsf{s}(t) \in \Pi_g^\circ$.
(ii) *The bezoutian* $B(t)$ *is positive definite.*

Proof. This is a particular case of a theorem of Sylvester, which states that the number of real roots of h_t is equal to $p - q$, where (p, q) is the signature of the real quadratic form

$$Q(x) = {}^t x.B(t).x,$$

where $x = (x_0, \dots, x_{g-1}) \in \mathbb{R}^g$. Here is a short proof: if $1 \leq j \leq g$, define the linear form

$$L_j(x) = x_0 + x_1 t_j + \cdots + x_{g-1} t_j^{g-1}.$$

Then $x.V(t) = (L_1(x), \dots, L_g(x))$, and

$$Q(x) = \sum_{j=1}^{g} L_j(x)^2.$$

If $t_j \in \mathbb{R}$, the linear form L_j is real. If $t_j \notin \mathbb{R}$, the non-real linear form $L_j = A_j + iB_j$ appears together with its conjugate, and

$$L_j^2 + \bar{L}_j^2 = 2A_j^2 - 2B_j^2.$$

This shows that if h_t has r real roots and s couples of non-real roots, the signature of Q is $(r + s, s)$. $\square$

The bezoutian $B(t)$ is positive definite if and only if its principal minors

$$M_j(p_1, \dots, p_g) = \begin{vmatrix} p_0 & p_1 & \dots & p_{j-1} \\ p_1 & p_2 & \dots & p_j \\ \dots & \dots & \dots & \dots \\ p_{j-1} & p_j & \dots & p_{2j-2} \end{vmatrix} \qquad (1 \leq j \leq g)$$

are > 0, see for instance [3, Prop. 3, p. 116]. By substituting to the power sums their expression given by Girard's formula of Proposition 6.1, we obtain g polynomials

$$m_j = M_j \circ \mathsf{v} \in \mathbb{Z}[s_1, \dots, s_g] \quad (1 \leq j \leq g).$$

Of course, $m_1 = g$, and

$$M_g(p_1, \dots, p_g) = \det B(t) = D_0(t),$$

hence, $m_g(s) = d_0(s)$. As a consequence of Lemma 6.3, we obtain

$$\Pi_g^\circ = \{s \in \mathbb{R}^g \mid m_j(s) > 0 \text{ if } 2 \leq j \leq g\},$$

hence:

Lemma 6.4. *We have*

$$\Pi_g = \{s \in \mathbb{R}^g \mid m_j(s) \geq 0 \textit{ if } 2 \leq j \leq g\}. \quad \square$$

Example 6.5. If $g = 2$, then $d_0(s) = s_1^2 - 4s_2$, and

$$\Pi_2 = \left\{s \in \mathbb{R}^2 \mid d_0(s) \geq 0\right\}.$$

Example 6.6. If $g = 3$, then

$$d_0(s) = s_1^2 s_2^2 - 4s_2^3 - 4s_1^3 s_3 + 18 s_1 s_2 s_3 - 27 s_3^2,$$

and $m_2(s) = 2(s_1^2 - 3s_2)$. But if $d_0(s) \geq 0$, then $m_2 \geq 0$. Actually, if

$$p = -\frac{s_1^2 - 3s_2}{3}, \quad q = \frac{2s_1^3 - 9s_1 s_2 + 27 s_3}{27},$$

then

$$d_0(s) = -(4p^3 + 27q^2), \quad m_2 = -6p.$$

If $d_0(s) \geq 0$, then $4p^3 \leq -27q^2$ and $p \leq 0$. Hence, as it is well known, Π_3 is defined by only one inequality:

$$\Pi_3 = \left\{ s \in \mathbb{R}^3 \mid d_0(s) \geq 0 \right\}.$$

7. The symmetric alcove

The *symmetric alcove* is the compact set

$$\Sigma_g = \mathsf{s}(I_g) \subset \Pi_g.$$

We have $\mathsf{s}(\bar{A}_g) = \mathsf{s}(I_g)$, and the induced map

$$I_g/\mathfrak{S}_g \xrightarrow{\sim} \Sigma_g$$

is a homeomorphism, leading to the commutative diagram

$$\begin{tikzcd}
 & I_g \arrow[dr, "\pi"] \arrow[dd, "\mathsf{s}"] & \\
\bar{A}_g \arrow[ur, "\iota"] \arrow[dr, "\simeq"'] & & I_g/\mathfrak{S}_g \arrow[dl, "\simeq"] \\
 & \Sigma_g &
\end{tikzcd}$$

If $p \in \mathbb{C}[t_1, \dots, t_g]$ is a symmetric polynomial and if $\lambda \in \mathbb{C}$, define

$$p(\lambda; t) = p(\lambda + t_1, \dots, \lambda + t_g), \quad t = (t_1, \dots, t_g).$$

The polynomial $p(\lambda; t)$ is symmetric with respect to t.

Lemma 7.1. *If $t \in \mathbb{R}^g$ and $\lambda > 0$, the following conditions are equivalent:*

(i) *$s_i(\lambda; t) > 0$ and $s_i(\lambda; -t) > 0$ for $1 \leq i \leq g$.*
(ii) *$|t_i| < \lambda$ for $1 \leq i \leq g$.*

Proof. It suffices to prove the following result: if $t \in \mathbb{R}^g$, the following conditions are equivalent:

(i) $s_i(t) > 0$ for $1 \leq i \leq g$.
(ii) $t_i > 0$ for $1 \leq i \leq g$.

If $h_t \in \mathbb{R}[u]$ is defined as in (6.1), namely

$$h_t(u) = (u - t_1) \dots (u - t_g) = u^g - s_1 u^{g-1} + \dots + (-1)^g s_g,$$

let $f^\flat(u) = (-1)^g f(-u)$. Then

$$f^\flat(u) = (u + t_1) \dots (u + t_g) = u^g + s_1 u^{g-1} + s_2 u^{g-2} + \dots + s_g,$$

and if (i) is satisfied, the roots of $(-1)^g f(-u)$ are < 0, and this implies (ii). The converse is trivial. $\square$

The polynomial $s_i(\lambda; t) \in \mathbb{C}[t_1, \dots, t_g]$ is a linear combination of elementary symmetric polynomials of degree $\leq i$:

Lemma 7.2. *If* $1 \le i \le g$ *and if* $\lambda > 0$*, then*

$$s_i(\lambda; t) = L_i^+(\lambda; \mathsf{s}(t)),$$

with

$$L_i^+(\lambda; s) = \sum_{k=0}^{i} \binom{g-i+k}{k} s_{i-k} \lambda^k,$$

which is a linear form with respect to $s_1, \ldots, s_i$*. Similarly,*

$$s_i(\lambda; -t) = L_i^-(\lambda; \mathsf{s}(t)),$$

where

$$L_i^-(\lambda; s_1, s_2, \ldots, s_i) = L_i^+(\lambda; -s_1, s_2, \ldots, (-1)^i s_i).$$

Proof. The Taylor expansion of $h_t(u-\lambda)$ with respect to u shows that

$$s_i(\lambda, t) = (-1)^i \, \frac{h_t^{(g-i)}(-\lambda)}{(g-i)!}\,.$$

The first formula is obtained by transferring these equalities in the Taylor expansion of $h_t^{(g-i)}(\lambda)$ at 0 :

$$h_t^{(g-i)}(-\lambda) = \sum_{k=0}^{i} (-1)^k h_t^{(g-i+k)}(0) \frac{\lambda^k}{k!}\,. \quad \square$$

The second formula is deduced from the first by noticing that $s_i(-t) = (-1)^i s_i(t)$.

Considering that $s_0 = 1$, we have

$$\begin{aligned} s_1(\lambda; t) &= s_1(t) + g\lambda, \\ s_2(\lambda; t) &= s_2(t) + (g-1)\lambda s_1(t) + \frac{g(g-1)}{2}\,\lambda^2, \\ s_g(\lambda; t) &= \sum_{k=0}^{g} s_{g-k}(t)\lambda^k = (-1)^g h_t(-\lambda) = \prod_{i=1}^{g} (t_i + \lambda). \end{aligned}$$

Hence

$$L_1^{\pm}(2; s) = \pm s_1 + 2g, \qquad L_2^{\pm}(2; s) = s_2 \pm 2(g-1)s_1 + 2g(g-1),$$

and

$$L_g^{\pm}(2; s) = \sum_{k=0}^{g} 2^k s_{g-k}, \qquad L_g^-(2; s) = \sum_{k=0}^{g} (-1)^{g-k} 2^k s_{g-k}. \tag{7.1}$$

From Lemmas 7.1 and 7.2 we obtain

Lemma 7.3. *Assume* $t \in \mathbb{R}^g$*. Then* $t \in I_g$ *if and only if* $\mathsf{s}(t) \in \Theta_g$*, where*

$$\Theta_g = \left\{ s \in \mathbb{R}^g \;\mid\; L_i^{\pm}(2; s) \ge 0 \textit{ for } 1 \le i \le g \right\}. \quad \square$$

Notice that the polyhedron Θ_g is unbounded.

Theorem 7.4. *If* $\Sigma_g = \mathsf{s}(I_g)$*, then*

$$\Sigma_g = \Theta_g \cap \Pi_g,$$

where Π_g and Θ_g are defined in Lemmas 6.4 and 7.3. Moreover, $\Sigma_g = \mathsf{s}(\bar{A}_g)$ is a semi-algebraic set homeomorphic to the g-dimensional simplex, and

$$\Sigma_g \subset \prod_{i=1}^{g} \left[-2^i \binom{g}{i}, 2^i \binom{g}{i} \right].$$

Proof. By definition, $\Pi_g = \mathsf{s}(\mathbb{R}^g)$, hence, the first statement follows from Lemma 7.3. The properties of Σ_g follow from Proposition 6.2, and the last statement is just a consequence of the definition of $s_1(t), \ldots, s_g(t)$. ☐

Example 7.5. If $g = 2$, then $d_0(s) = s_1^2 - 4s_2$, and

$$\Pi_2 = \left\{ s \in \mathbb{R}^2 \mid d_0(s) \geq 0 \right\},$$

as we saw in Example 6.5. The triangle Θ_2 is defined by four inequalities:

$$L_1^{\pm}(2, s) = \pm s_1 + 4 \geq 0,$$
$$L_2^{\pm}(2, s) = s_2 \pm 2s_1 + 4 \geq 0.$$

The symmetric alcove Σ_2 the curvilinear triangle, drawn in Figure 5, contained in the square $[-4, 4] \times [-4, 4]$.

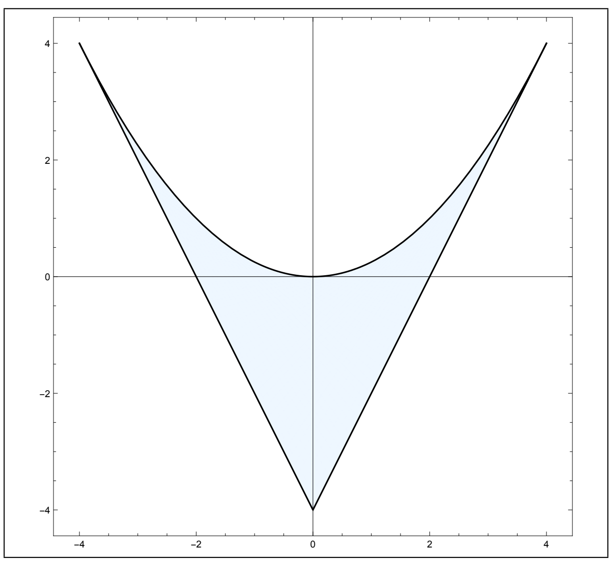

FIGURE 5. The symmetric alcove Σ_2.

Example 7.6. If $g = 3$, then

$$d_0(s) = s_1^2 s_2^2 - 4s_2^3 - 4s_1^3 s_3 + 18 s_1 s_2 s_3 - 27 s_3^2,$$

and

$$\Pi_3 = \left\{ s \in \mathbb{R}^3 \mid d_0(s) \geq 0 \right\},$$

as we saw in Example 6.6. The polyhedron Θ_3 is defined by six inequalities

$$L_1^{\pm}(2,s) = \pm s_1 + 6 \geq 0, \qquad L_2^{\pm}(2,s) = s_2 \pm 4s_1 + 12 \geq 0,$$
$$L_3^{\pm}(2,s) = \pm s_3 + 2s_2 \pm 4s_1 + 8 \geq 0.$$

The intersection of Θ_3 and of the box

$$[-4,4] \times [-12,12] \times [-8,8]$$

make up a polytope P_3 with 6 vertices

$$\begin{array}{llll} p_1 = (-6,12,-8) & = \mathsf{s}(-2,-2,-2), & p_2 = (-2,-4,8) & = \mathsf{s}(2,-2,-2), \\ p_3 = (2,-4,-8) & = \mathsf{s}(2,2,-2), & p_4 = (6,12,8) & = \mathsf{s}(2,2,2), \\ p_5 = (6,12,-8), & & p_6 = (-6,12,8), & \end{array}$$

and 7 facets supported the following hyperplanes:

$$s_2 = 12, \quad s_3 = \pm 8, \quad L_2^{\pm}(s) = 0, \quad L_3^{\pm}(s) = 0.$$

Then

$$\Sigma_3 = \Pi_3 \cap P_3.$$

The symmetric alcove Σ_3 is drawn in Figure 6. This set is invariant by the symmetry $(s_1, s_2, s_3) \mapsto (-s_1, s_2, -s_3)$. The graphical representation leads to suppose that

$$\Sigma_3 = \Pi_3 \cap \Delta_3,$$

where Δ_3 is the tetrahedron with vertices p_1, p_2, p_3, p_4 and support hyperplanes

$$L_3^{\pm}(s) = 0, \quad L_0^{\pm}(s) = 0,$$

where $L_0^{\pm}(s) = 24 \pm 4s_1 - 2s_2 \mp 3s_3$.

8. Symmetric integration formula

The map $\varphi \mapsto \varphi \circ \mathsf{s}$ defines an isomorphism

$$\mathsf{s}^* : \mathscr{C}(\Sigma_g) \xrightarrow{\ \sim\ } \mathscr{C}(I_g)^{\mathrm{sym}} = \mathscr{C}(\bar{A}_g)$$

If $\mathsf{F} \in \mathscr{C}(G)^\circ$, we denote by $\widetilde{\mathsf{F}}$ the unique function in $\mathscr{C}(I_g)^{\mathrm{sym}}$ such that

$$\widetilde{\mathsf{F}} \circ \mathsf{s}(t) = \mathsf{F} \circ k(t),$$

that is,

$$\widetilde{\mathsf{F}} \circ \mathsf{s}(2\cos\theta_1, \dots, 2\cos\theta_g) = \mathsf{F} \circ h(\theta_1, \dots, \theta_g).$$

Then the map $\mathsf{F} \mapsto \widetilde{\mathsf{F}}$ is an isomorphism

$$\mathscr{C}(G)^\circ \xrightarrow{\ \sim\ } \mathscr{C}(\Sigma_g).$$

inducing by restriction an isomorphism, cf. Proposition A.1 in the appendix:

$$R(G) \xrightarrow{\ \sim\ } \mathbb{Z}[s_1, \dots, s_g].$$

Since $D_1 \in \mathbb{Z}[t_1, \dots, t_g]^{\mathrm{sym}}$, there is a polynomial $d_1 \in \mathbb{Z}[s_1, \dots, s_g]$ such that

$$d_1(\mathsf{s}(t)) = D_1(t) = \prod_{j=1}^{g} (4 - t_j^2), \tag{8.1}$$

hence

$$D_1(t) = s_g(2;t) s_g(2;-t),$$

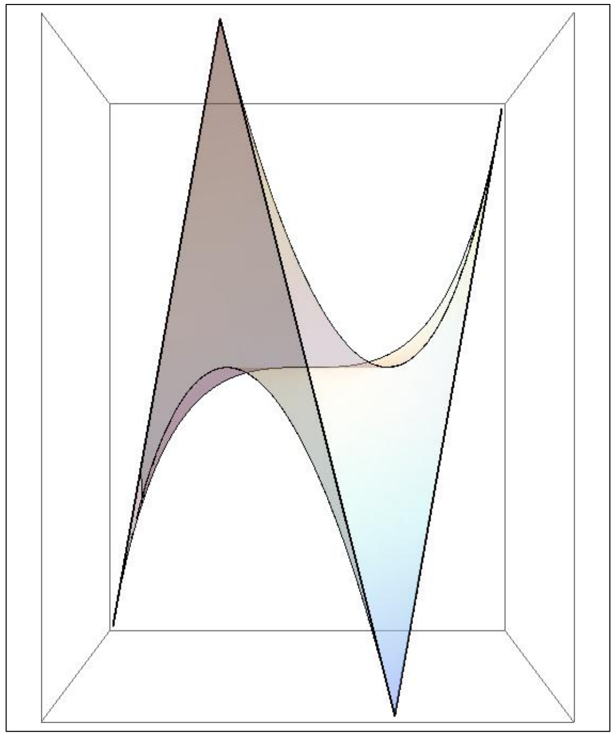

FIGURE 6. The symmetric alcove Σ_3.

where $s_\pm(\lambda; t)$ is defined in Lemma 7.2, and

$$d_1(s) = L_g^+(2; s) L_g^-(2; s),$$

where $L_g^\pm(2; s)$ is defined by (7.1).

Proposition 8.1 (Symmetric integration formula)**.** *If* $\mathsf{F} \in \mathscr{C}(G)^\circ$, *then*

$$\int_G \mathsf{F}(m)\, dm = \int_{\Sigma_g} \widetilde{\mathsf{F}}(s) \nu_g(s)\, ds,$$

with $ds = ds_1 \dots ds_g$, *and*

$$\nu_g(s) = \frac{1}{(2\pi)^g} \sqrt{d_0(s) d_1(s)},$$

where $d_0(s)$ *is given by* (6.3) *and* $d_1(s)$ *by* (8.1).

Proof. By Proposition 6.2, we can perform a change of variables from Σ_g to A_g, apart from null sets, by putting $s = \mathsf{s}(t)$. If $\varphi \in \mathscr{C}(\Sigma_g)$, we have

$$\int_{\Sigma_g} \varphi(s) \frac{ds}{\sqrt{d_0(s)}} = \int_{A_g} \varphi(\mathsf{s}(t))\, dt.$$

This implies

$$\int_{\Sigma_g} \varphi(s) \nu_g(s)\, ds = g! \int_{A_g} \varphi(\mathsf{s}(t)) \lambda_g(t).$$

If $\varphi = \widetilde{\mathsf{F}}$, then

$$\int_{\Sigma_g} \widetilde{\mathsf{F}}(s)\nu_g(s)\,ds = g!\int_{A_g} \mathsf{F}\circ k(t)\lambda_g(t)\,dt = \int_{I_g} \mathsf{F}\circ k(t)\lambda_g(t)\,dt,$$

the second equality by using (3.5). One concludes with the help of Proposition 3.2. □

In other words, if ϕ_g is the characteristic function of Σ_g, the function $\nu_g\,\phi_g$ is the joint probability distribution density function of the distribution for the random variables $s_1,\dots,s_g$.

If $\boldsymbol{\tau}$ is the trace map on $\mathbf{USp}_{2g}$, then

$$\widetilde{\boldsymbol{\tau}}(s) = s_1.$$

One obtains an integral expression of the density by the method of integration along the fibers already used in Remark 3.5, which reduces here to an application of Fubini's theorem. The linear form $s \mapsto s_1$ is a submersion from the open dense subset $U = \mathsf{s}(A_g)$ of Σ_g onto $J = (-2g, 2g)$, and if $x \in J$, then

$$V_x = \{s \in U \mid s_1 = x\}$$

is just an intersection with a hyperplane. If

$$\alpha_x(s_2,\dots,s_g) = \nu_g(x,s_2,\dots,s_g)\,ds_2\wedge\cdots\wedge ds_g,$$

then

$$\Phi_{\boldsymbol{\tau}}(z) = \int_{s_1\le z} \nu_g(s)ds_1\wedge\cdots\wedge ds_g = \int_{-2g}^{z} dx\int_{V_s}\alpha_x(s_2,\dots,s_g),$$

hence:

Proposition 8.2. *If $|x| < 2g$, then*

$$f_{\boldsymbol{\tau}}(x) = \int_{V_x}\alpha_x(s_2,\dots,s_g). \quad \square$$

Example 8.3. If $g = 2$, the symmetric alcove Σ_2 is described in Example 7.5. Here,

$$d_0(s) = s_1^2 - 4s_2, \quad d_1(s) = (s_2+4)^2 - 4s_1^2, \quad \nu_2(s) = \frac{1}{4\pi^2}\sqrt{d_0(s)d_1(s)}.$$

The graph of ν_2 is shown in Figure 7. The maximum of ν_2 in Σ_2 is attained at the point

$$s_0 = (0, -\frac{4}{3}), \quad \text{with} \quad \nu_2(s_0) = \frac{8}{3\sqrt{3}\pi^2} = 0.155\dots$$

By Proposition 8.2, we find a definite integral : if $x \ge 0$,

$$f_{\boldsymbol{\tau}}(x) = \frac{1}{4\pi^2}\int_{2x-4}^{x^2/4}\left[\left((y+4)^2 - 4x^2\right)\left(x^2-4y\right)\right]^{\frac{1}{2}}dy.$$

It can be verified that this formula is in accordance with Theorem 5.4.

Example 8.4. If $g = 3$, the symmetric alcove Σ_3 is described in Example 7.6. Here,

$$\begin{aligned} d_0(s) &= s_1^2s_2^2 - 4s_2^3 - 4s_1^3s_3 + 18s_1s_2s_3 - 27s_3^2,\\ d_1(s) &= (2s_2+8)^2 - (4s_1+s_3)^2,\\ \nu_3(s) &= \frac{1}{8\pi^3}\sqrt{d_0(s)d_1(s)}, \end{aligned}$$

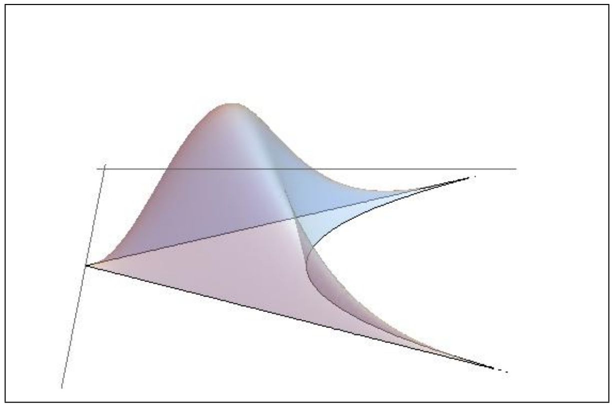

FIGURE 7. The density ν_2.

and

$$V_x = \left\{ s \in \mathbb{R}^3 \mid d_0(s) \geq 0,\ \pm s_3 + 2s_2 \pm 4x + 8 \geq 0 \right\}.$$

The density is

$$f_{\boldsymbol{\tau}}(x) = \int_{V_x} \nu_3(x, s_2, s_3) ds_2 ds_3.$$

With this formula in hands, we are able to compute the even moments :

$$1, 1, 3, 15, 104, 909, 9\,449, 112\,398, 1\,489\,410, 21\,562\,086 \ldots$$

This sequence is in accordance with the results of [12, Sec. 4] and the sequence A138540 in the OEIS [16]. Actually, it is faster to compute this sequence by noticing that, according to Weyl's integration formula of Proposition 3.2, the characteristic function of $\boldsymbol{\tau}$ is given, for $y \in \mathbb{R}$, by

$$\widehat{f_{\boldsymbol{\tau}}}(y) = \frac{1}{8\pi^3} \int_{I_3} D_0(t) \sqrt{D_1(t)} \cos(y(t_1 + t_2 + t_3))\, dt_1 dt_2 dt_3.$$

An implementation of this integral in the *Mathematica* software gives

$$\widehat{f_{\boldsymbol{\tau}}}(y) = \\ 24 \left(-\frac{4J_1(2y)^3}{y^5} + \frac{11 J_1(2y)^2 J_2(2y)}{y^6} - \frac{2(3 + y^2) J_1(2y) J_2(2y)^2}{y^7} + \frac{5 J_2(2y)^3}{y^6} \right),$$

and it suffices to apply (3.6) to obtain the moments. An approximation of $f_{\boldsymbol{\tau}}$ to any order in $L^2([-6, 6], dx)$ can be obtained from the sequence of moments, using Legendre polynomials, which form an orthogonal basis of $L^2([-1, 1], dx)$. For instance the maximum of $f_{\boldsymbol{\tau}}$ is reached for $x = 0$, and we find

$$f_{\boldsymbol{\tau}}(0) = 0.396\,467 \ldots$$

The graph of $f_{\boldsymbol{\tau}}$ obtained by this approximation process is drawn in Figure 8.

As a final instance, we come back to the case $g = 2$ and apply the symmetric integration formula to the distribution of the character $\boldsymbol{\tau}_2$ of the exterior power $\wedge^2 \pi$ of the identity representation π of $\mathbf{USp}_4$ on $\mathbb{C}^4$, namely

$$\boldsymbol{\tau}_2 \circ h(\theta) = 2 + 4 \cos\theta_1 \cos\theta_2, \qquad \widetilde{\boldsymbol{\tau}_2}(s) = s_2 + 2.$$

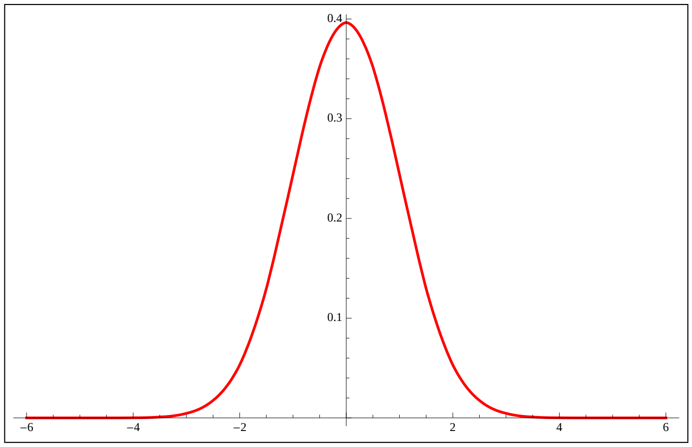

FIGURE 8. Density of the distribution of $\boldsymbol{\tau}$, case $g = 3$

The density of $\boldsymbol{\tau}_2$ is given by

$$f_{\boldsymbol{\tau}_2}(x) = \int_{I_\pm} \sqrt{16 + 8x + x^2 - 4z^2}\sqrt{z^2 - 4x}\; dz,$$

with

$$I_- = \left(-\frac{x+4}{2}, \frac{x+4}{2}\right) \qquad \text{if } -4 < x < 0,$$

$$I_+ = \left(-\frac{x+4}{2}, -2\sqrt{x}\right) \cup \left(2\sqrt{x}, \frac{x+4}{2}\right) \qquad \text{if } 0 < x < 4.$$

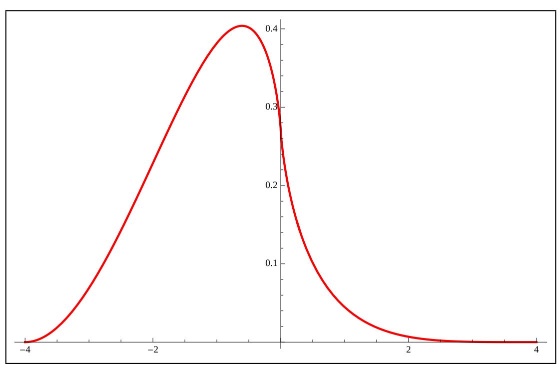

FIGURE 9. Density of the distribution of $\boldsymbol{\tau}_2$, case $g = 2$

The implementation of this integral in the *Mathematica* software gives the sequence of moments (see below), from which one deduces:

Proposition 8.5. *Assume $|x| < 4$. Then $f_{\boldsymbol{\tau}_2}(x)$ is equal to*

$$\frac{\operatorname{sgn}(x)}{24\pi^2}\left(x(x^2 - 24x + 16)E\left(1 - \frac{16}{x^2}\right) + 4(3x^2 - 8x + 48)K\left(1 - \frac{16}{x^2}\right)\right). \qquad \square$$

The maximum of $f_{\boldsymbol{\tau}_2}$ is reached for $x_0 = -0.605\ldots$, and $f_{\boldsymbol{\tau}_2}(x_0) = 0.403\ldots$ Moreover

$$f_{\boldsymbol{\tau}_2}(0) = \frac{8}{3\pi^2} = 0.270\ldots$$

This function is continuous, but the derivative has a logarithmic singularity:

$$f'_{\boldsymbol{\tau}_2}(x) \sim \frac{\log x^2}{\pi^2}, \quad x \to 0.$$

The graph of $f_{\boldsymbol{\tau}_2}$ is shown in Figure 9. The moments M_n of $f_{\boldsymbol{\tau}_2}$ are obtained by numerical integration:

$$1, -1, 2, -4, 10, -25, 70, -196, 588, -1764\ldots$$

Hence, the random variable $\boldsymbol{\tau}_2$ has mean -1 and variance 2. This sequence is, up to sign, the sequence A005817 in the OEIS [16], such that

$$M_{2n}(\boldsymbol{\tau}_2) = C_n C_{n+1}, \quad M_{2n+1} = -C_{n+1}^2,$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is the nth *Catalan number*.

Remark 8.6. The representation $\wedge^2\pi$ is reducible, cf. for instance Lemma A.6 in the appendix. Actually, the two fundamental representations of $\mathbf{USp}_2$ are the identity representation $\pi = \pi_1$ with character $\boldsymbol{\tau}$ and a representation π_2 of dimension 5 and character $\boldsymbol{\chi}_2$ satisfying

$$\boldsymbol{\chi}_2 \circ h(\theta) = 1 + 4\cos\theta_1\cos\theta_2, \qquad \widetilde{\boldsymbol{\chi}_2}(s) = s_2 + 1.$$

The representation π_2 is equivalent to the representation corresponding to the morphism of $\mathbf{USp}_2$ onto $\mathbf{SO}_5$. Since $\boldsymbol{\tau}_2 = \boldsymbol{\chi}_2 + 1$, we have

$$\wedge^2\pi = \pi_2 \oplus 1,$$

and $\widetilde{\boldsymbol{\tau}_2}(s) = s_2 + 2$: this relation is an instance of Theorem A.7 in the appendix. The random variable $\boldsymbol{\chi}_2$, with values on $[-3, 5]$, is standardized, by Remark 3.4. The moments of $f_{\boldsymbol{\chi}_2}$ are

$$0, 1, 0, 3, 1, 15, 15, 105, 190, 945\ldots$$

in accordance with the sequence A095922 in the OEIS [16].

Appendix A. The character ring of G

The *character ring* $R(G)$ of $G = \mathbf{USp}_{2g}$ is the subring of $\mathscr{C}(G)^\circ$ generated, as a $\mathbb{Z}$-module, by the characters of continuous representations of G on finite dimensional complex vector spaces. Since every representation of G is semi-simple, the $\mathbb{Z}$-module $R(G)$ is free and admits as a basis the set $\widehat{G}$ of characters of irreducible representations of G :

$$R(G) = \sum_{\tau\in\widehat{G}} \mathbb{Z}\tau.$$

The *virtual characters* are the elements of $R(G)$, and the characters correspond to the additive submonoid of sums over $\widehat{G}$ with non-negative coefficients. The functions $\theta \mapsto e^{i\theta_j}(1 \le i \le g)$ make up a basis of the discrete group $\mathbb{T}^g$, and if $h(\theta)$ is as in (2.1), the map $h^* : f \mapsto f \circ h$ defines an isomorphism from the group $\mathsf{X}(T)$

of characters of T to the group $\mathsf{X}(\mathbb{T}^g)$. Hence, if $\mathbb{Z}[\mathsf{X}(T)]$ is the group ring, we have a ring isomorphism

$$h^* : \mathbb{Z}[\mathsf{X}(T)] \xrightarrow{\sim} \mathbb{Z}[\{e^{i\theta_j}, e^{-i\theta_j}\}].$$

Let $\mathbb{Z}[\mathsf{X}(T)]^W$ be the subring of elements invariants under W. Recall that the restriction map

$$R(G) \xrightarrow{\sim} \mathbb{Z}[\mathsf{X}(T)]^W$$

is a ring isomorphism [4, Ch. 9, §7, n°3, Cor., p. 353]. From the structure of W, we deduce that h^* induces a ring isomorphism

$$R(G) \xrightarrow{\sim} \mathbb{Z}[2\cos\theta_1, \ldots 2\cos\theta_g]^{\mathrm{sym}}. \tag{A.1}$$

and, by putting $(2\cos\theta_1, \ldots 2\cos\theta_g) = (t_1, \ldots, t_g)$, a ring isomorphism

$$R(G) \xrightarrow{\sim} \mathbb{Z}[t_1, \ldots t_g]^{\mathrm{sym}}. \tag{A.2}$$

On the other hand, the application $\varphi \mapsto \varphi \circ \mathsf{s}$, where s is the Viète map, induces the classical isomorphism

$$\mathsf{s}^* : \mathbb{Z}[s_1, \ldots, s_g] \xrightarrow{\sim} \mathbb{Z}[t_1, \ldots, t_g]^{\mathrm{sym}}.$$

If $\mathsf{F} \in R(G)$, we denote by $\widetilde{\mathsf{F}}$ the unique polynomial in $\mathbb{Z}[s_1, \ldots, s_g]$ such that

$$\widetilde{\mathsf{F}} \circ \mathsf{s}(2\cos\theta_1, \ldots, 2\cos\theta_g) = \mathsf{F} \circ h(\theta_1, \ldots, \theta_g).$$

We obtain:

Proposition A.1. *If $G = \mathbf{USp}_{2g}$, the map $\mathsf{F} \mapsto \widetilde{\mathsf{F}}$ is a ring isomorphism*

$$R(G) \xrightarrow{\sim} \mathbb{Z}[s_1, \ldots, s_g]. \quad \square$$

Recall from Remark 2.1 that Φ_{2g} is the set of monic palindromic polynomials of degree $2g$ in $\mathbb{C}[u]$ with all roots on the unit circle. We write a typical element of Φ_{2g} as

$$p_a(u) = \sum_{n=0}^{2g} (-1)^n a_n u^{2g-n},$$

where $a_{2g-n} = a_n$ for $0 \le n \le g$. Moreover $p_a(u) = u^{2g} p_a(u^{-1})$. The roots of p_a come by pairs : if p_a is monic, then

$$p_a(u) = \prod_{j=1}^{g} (u - e^{i\theta_j})(u - e^{-i\theta_j}) = \prod_{j=1}^{g} (u^2 - ut_j + 1),$$

with $t_j = 2\cos\theta_j$, and the coefficients a_n are symmetric polynomials in the variables $\{e^{i\theta_j}, e^{-i\theta_j}\}$, invariant under conjugation.

Theorem A.2. *If $t = (t_1, \ldots, t_g) \in \mathbb{C}^g$, and if $0 \le n \le 2g$, define $a_n(t)$ by the relation*

$$\prod_{j=1}^{g} (u^2 - ut_j + 1) = \sum_{n=0}^{2g} (-1)^n a_n(t) u^{2g-n}.$$

If $0 \le n \le g$, then

$$a_n(t) = \sum_{j=0}^{n/2} \binom{g + 2j - n}{j} s_{n-2j}(t),$$

where $s_0(t) = 1$ and $s_n(t)$ is the elementary symmetric polynomial of degree n.

We deduce Theorem A.2 from the following lemma.

Lemma A.3. *If* $s = (s_0, \ldots, s_g) \in \mathbb{C}^{g+1}$, *let*

$$h_s(u) = \sum_{n=0}^{g} (-1)^n s_n u^{g-n},$$

and for $0 \le n \le 2g$, *define* $q_n(s)$ *by the relation*

$$u^g h_s(u + u^{-1}) = \sum_{n=0}^{2g} (-1)^n q_n(s) u^{2g-n}.$$

If $0 \le n \le g$, *then*

$$q_n(s) = \sum_{j=0}^{n/2} \binom{g+2j-n}{j} s_{n-2j}.$$

Proof. We have

$$u^g h(u+u^{-1}) = u^g \sum_{k=0}^{g} (-1)^k s_k (u+u^{-1})^{g-k}.$$

Since

$$u^g (u+u^{-1})^{g-k} = u^g \sum_{j=0}^{g-k} \binom{g-k}{j} (u^{-1})^{g-k-j} u^j = \sum_{j=0}^{g-k} \binom{g-k}{j} u^{k+2j},$$

one finds

$$u^g h(u+u^{-1}) = \sum_{j+k \le g, j \ge 0, k \ge 0} (-1)^k \binom{g-k}{j} s_k u^{k+2j}.$$

Let $k + 2j = n$. Then n runs over the full interval $[0, 2g]$ and

$$j \ge 0 \text{ and } k \ge 0 \text{ and } j + k \le g \quad \Longleftrightarrow \quad j \ge 0 \text{ and } 2j \le n \text{ and } j \ge n - g.$$

Hence, if $1 \le n \le 2g$, we have

$$q_n(s) = \sum_{j=\max(0,n-g)}^{n/2} \binom{g+2j-n}{j} s_{n-2j},$$

and the result follows. □

If p is a Weil polynomial and if $p(u) = u^g h(u+u^{-1})$, then h has real roots and is called the *real Weil polynomial* associated to p.

Proof of Theorem A.2. In Lemma A.3, assume that

$$h_s(u) = \prod_{j=1}^{g} (u - t_j).$$

Then $s_n = s_n(t)$, where $t = (t_1, \ldots, t_g)$, and

$$u^g h_s(u+u^{-1}) = \prod_{j=1}^{g} (u^2 - ut_j + 1).$$

Hence, $a_n(t) = q_n(1, s_1(t), \ldots, s_n(t))$. □

Define an endomorphism q of $\mathbb{C}^{g+1}$ by

$$\mathsf{q} : (s_0, \ldots, s_g) \mapsto (q_0(s), \ldots, q_g(s)).$$

By Lemma A.3, the square matrix of order $g+1$ associated to q is unipotent and lower triangular, with coefficients in $\mathbb{N}$. For instance:

$$\text{if } g = 2 : \mathsf{q} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}; \quad \text{if } g = 3 : \mathsf{q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}. \tag{A.3}$$

Remark A.4. A reciprocal of the map $\mathbf{Q} : h \mapsto u^g h(u + u^{-1})$ is constructed as follows. If $n \geq 1$, let $T_n(u)$ be the n-th *Chebyshev polynomial* [9, p. 993], and $c_n(u) = 2T_n(u/2)$, in such a way that

$$c_n(u + u^{-1}) = u^n + u^{-n} \quad \text{if } n \geq 1.$$

Moreover put $c_0(u) = 1$. If $p_a \in \Phi_{2g}$ as above and if

$$[\mathbf{R}p](u) = \sum_{n=0}^{g} (-1)^n a_n c_{g-n}(u),$$

it is easy to see that $\mathbf{Q} \circ \mathbf{R}(p) = p$.

Remark A.5. It is worthwile to notice that the map $\varphi \mapsto \varphi \circ \mathsf{q}$ defines an isomorphism of the two ring of invariants:

$$\mathsf{q}^* : \mathbb{Z}[a_1, \ldots, a_g] \xrightarrow{\sim} \mathbb{Z}[s_1, \ldots, s_g]$$

where we identify $\mathbb{Z}[a_1, \ldots, a_g]$ and $\mathbb{Z}[a_1, \ldots, a_{2g-1}]/((a_{2g-n} - a_n))$. If we define a polynomial mapping $\mathsf{a} : \mathbb{C}^g \longrightarrow \mathbb{C}^{g+1}$ by

$$\mathsf{a} : t = (t_1, \ldots, t_g) \mapsto (a_0(t), \ldots, a_g(t)).$$

and if $\mathsf{s}(t) = (s_0(t), s_1(t), \ldots, s_g(t))$ is the (extended) Viète map, then

$$\mathsf{a} = \mathsf{q} \circ \mathsf{s}.$$

These maps are gathered in the following diagram:

$$\begin{tikzcd} & \mathbb{Z}[\{e^{i\theta_j}, e^{-i\theta_j}\}] \arrow[rr] \arrow[d, dash, "W"] & & \mathbb{Z}[t_1, \ldots, t_g] \arrow[d, dash, "\mathfrak{S}_g"] \\ R(G) : & \mathbb{Z}[a_1, \ldots, a_g] \arrow[r, "\mathsf{q}^*"] \arrow[rr, bend right, "\mathsf{a}^*"'] & \mathbb{Z}[s_1, \ldots, s_g] \arrow[r, "\mathsf{s}^*"] & \mathbb{Z}[t_1, \ldots, t_g]^{\mathrm{sym}} \end{tikzcd}$$

We apply the preceding to the character of the n-th exterior power $\wedge^n \pi$ of the identity representation π of G in $\mathbb{C}^{2g}$. For $0 \leq n \leq 2g$, let

$$\boldsymbol{\tau}_n(m) = \mathrm{Trace}(\wedge^n m). \tag{A.4}$$

Generally speaking, the representation $\wedge^n \pi$ is reducible, and we describe now its decomposition. For each dominant weight ω of $\mathbf{USp}_{2g}$, we denote by $\pi(\omega)$ the irreducible representation with highest weight ω, cf. [4]. The following lemma is used in Lemma 4.4.

Lemma A.6. *Let $\omega_1, \ldots, \omega_g$ be the fundamental weights of $\mathbf{USp}_{2g}$. Then*

(i) *If* $1 \leq 2n+1 \leq g$, *we have*

$$\wedge^{2n+1}\pi = \bigoplus_{0\leq j\leq n} \pi(\omega_{2j+1}).$$

(ii) *If* $2 \leq 2n \leq g$, *we have*

$$\wedge^{2n}\pi = 1 \oplus \bigoplus_{1\leq j\leq n} \pi(\omega_{2j}).$$

Proof. See [11, Lemma, p. 62]; the corresponding result for a simple Lie algebra of type C_g is proved in [4, Ch. 8, §13, n°3, (IV), p. 206-209]. □

The characteristic polynomial of $m \in \mathbf{USp}_{2g}$ is

$$\mathrm{cp}_m(u) = \det(u.\mathbf{I} - m) = \sum_{n=0}^{2g} (-1)^n \boldsymbol{\tau}_n(m) u^{2g-n}.$$

The dual pairing

$$\wedge^n V \times \wedge^{2g-n} V \longrightarrow \wedge^{2g} V = \mathbb{C}$$

implies that $\boldsymbol{\tau}_{2g-n} = \boldsymbol{\tau}_n$ for $0 \leq n \leq 2g$, and this proves that $\mathrm{cp}_m \in \Phi_{2g}$. If m is conjugate to $h(\theta_1, \ldots, \theta_g)$, then

$$\mathrm{cp}_m(u) = \prod_{j=1}^{g} (u^2 - ut_j + 1),$$

with $t_j = 2\cos\theta_j$, and hence $\boldsymbol{\tau}_n \circ k \in \mathbb{Z}[t_1, \ldots t_g]^{\mathrm{sym}}$ as expected.

In the notation of Theorem A.2, we have

$$\boldsymbol{\tau}_n \circ k(t) = a_n(t), \tag{A.5}$$

and we deduce from this theorem:

Theorem A.7. *Let* $m \in \mathbf{USp}_{2g}$ *be conjugate to* $k(t_1, \ldots, t_g)$. *If* $0 \leq n \leq g$, *then*

$$\boldsymbol{\tau}_n(m) = \sum_{j=0}^{n/2} \binom{g+2j-n}{j} s_{n-2j}(t). \quad \square$$

For instance, according to (A.3):

— if $g = 2$: $\boldsymbol{\tau}_2(m) = s_2(t) + 2$ (cf. Remark 8.6),
— if $g = 3$: $\boldsymbol{\tau}_2(m) = s_2(t) + 3$, $\boldsymbol{\tau}_3(m) = s_3(t) + 2s_1(t)$.

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